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Multiple Lyapunov functions approach to observer-based H_∞ control for switched systems

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This article investigates the issue of H_∞ control for a class of continuous-time switched Lipschitz nonlinear systems. None of the individual subsystems is assumed to be stabilisable with H_∞ disturbance attenuation. Based on a generalised multiple Lyapunov functions (GMLFs) approach, which removes the nonincreasing requirement at switching points, a sufficient condition for the solvability of the H_∞ control problem under a state estimation-dependent switching law is presented. Observers, controllers and a switching law are simultaneously designed. As an extension, a sufficient condition for exponential stabilisability is also given.

Keywords: switched systems; observer; H_∞ control; exponential stabilisation; multiple Lyapunov functions

1. Introduction

Recent years have witnessed an enormous growth of interest in switched systems (Peleties and DeCarlo 1991; Branicky 1998; Liberzon 2003; Zhao and Hill 2008). The multiple Lyapunov functions technique (Peleties and DeCarlo 1991; Branicky 1998; Liberzon 2003) has been proved as a powerful and effective tool with less conservativeness. The key point of these results is that any Lyapunov function is nonincreasing over the 'switching on' time sequence of the corresponding subsystems, which is usually hard to check and difficult to satisfy. Thus, the 'min-switching' strategy (Liberzon 2003), connecting adjacent Lyapunov functions at switching points, becomes a widely accepted strategy which is a special case of Branicky (1998), but easy to design and realise. In fact, multiple Lyapunov functions are not necessarily connected to each other at switching points. A generalised multiple Lyapunov functins (GMLFs) method recently addressed in Zhao and Hill (2008) can overcome this problem and allow the 'jump' of adjacent Lyapunov functions at switching points when the system state can be observed.

On the other hand, the information of the state variable is usually unavailable or not fully available in engineering practice, and the state estimation can be used for control, diagnosis or supervision purposes. Inspired by these facts, for switched systems, the observer-based control problems are also important issues for both theoretical investigation as well as practical applications (Li, Wen, and Soh 2003; Rodrigues and How 2003; Ji, Wang, Xie, and Hao 2004; Xie, Xu, and Chen 2008). Such a design problem usually involves observer design (Alessandri and Coletta 2001; Bara, Daafouz, Kratz, and Ragot 2001; Pettersson 2005; Juloski, Heemels, and Weiland 2007), controller design (Feng 2002a,b; Chen, Zhu, and Feng 2004; de Best, Bukkems, van de Molengraft, Heemels, and Steinbuch 2008; van de Wouw and Pavlov 2008; Wang, Zhao, and Dimirovski 2009) and switching law design (Colaneri, Geromel, and Astolfi 2008; Xiang and Xiao 2011). Ji et al. (2004) derived quadratic stabilisation condition for switched linear systems via single Lyapunov function approach. Xie et al. (2008) studied the output stabilisability and observer-based switched control design problems of switched linear systems at any given switching frequency. Recently, Xiang and Xiao (2011) provided a discussion on constructing a switching law determined by the state variable of a full-order linear switched filter. However, to the best of the authors' knowledge, the observer-based control problems based on the GMLFs approach for switched systems have not been investigated yet.

This article deals with the observer-based H_∞ control by using the GMLFs approach for a class of switched Lipschitz nonlinear systems. Compared with the existing results, the observer-based GMLFs approach is employed to solve the H_∞ control problem when only an estimate of the state rather than the state

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is assumed to be available for designing the switching strategy and controllers. Secondly, the differentiable Lipschitz nonlinearity allows large values of the Lipschitz constant to be compared with the classical ones. Thirdly, exponential stabilisation is achieved while the existing works usually address asymptotical stabilisation. Besides, none of the individual subsystem is assumed to be stabilisable due to its significance both in theory and engineering application (see Liberzon (2003) and references therein).

Throughout this article, $\bar{\lambda}(\cdot)(\underline{\lambda}(\cdot))$ denotes the largest (smallest) eigenvalue of a symmetric matrix. $Co(a,b)=\{\lambda a+(1-\lambda)b,\ 0\le\lambda\le 1\}$ is the convex hull of a, b. $e_s(i)=(\underbrace{0,\dots,0,\overbrace{1}^{i\text{th}},0,\dots,0}_{s\,\text{components}})^{\mathrm{T}}$ are vectors of the canonical basis of $\Re^s$ for all $s\ge 1$.

2. Preliminaries

Consider the class of switched nonlinear systems:

$$\begin{aligned}\dot{x}(t)&=A_\sigma x(t)+B_\sigma u_\sigma+D_\sigma f_\sigma(x(t),y(t),u_\sigma)+W_\sigma\omega(t),\\ y(t)&=g_\sigma(x(t),u_\sigma),\\ z(t)&=\begin{bmatrix}E_\sigma x(t)\\ u_\sigma\end{bmatrix},\end{aligned}\tag{1}$$

where $\sigma:\Re^+\mapsto M=\{1,2,\dots,m\}$ is the right continuous piecewise constant switching signal to be designed, $x\in\Re^n$ is the state vector, $u_i\in\Re^{m_i}$ are the control inputs, $\omega\in\Re^{h_i}$ which belongs to $L_2[0,\infty)$ denotes the disturbance input, $y\in\Re^{p_i}$ and $z\in\Re^{r_i}$ denote the measured output and controlled output, respectively, A_i, B_i, D_i, E_i and W_i are constant matrices of appropriate dimensions.

Assumption 1: *The nonlinear functions $f_i:\Re^n\times\Re^{p_i}\times\Re^{m_i}\mapsto\Re^{q_i}$ and $g_i:\Re^n\times\Re^{m_i}\mapsto\Re^{p_i}$ are differentiable with respect to x, and*

$$\underline{f}^i_{jk}\le\frac{\partial f_{ij}}{\partial x_k}(x,y,u_i)\le\bar{f}^i_{jk},\quad \underline{g}^i_{jk}\le\frac{\partial g_{ij}}{\partial x_k}(x,u_i)\le\bar{g}^i_{jk},$$

f_{ij}, g_{ij} and x_j denote the j-th components of f_i, g_i and x, respectively, and $f_i(0,y,u_i)\equiv 0$.

Consider the following standard state observers:

$$\begin{aligned}\dot{\hat{x}}(t)&=A_\sigma\hat{x}(t)+B_\sigma u_\sigma+D_\sigma f_\sigma(\hat{x}(t),y(t),u_\sigma)\\ &\quad-L_\sigma(g_\sigma(\hat{x},u_\sigma)-g_\sigma(x,u_\sigma)),\end{aligned}\tag{2}$$

where $\hat{x}(t)$ denotes the estimate of the state $x(t)$ and the observer gain matrices $L_i\in\Re^{n\times p_i}$ will be determined later. The estimation error $e(t)=\hat{x}(t)-x(t)$ satisfies

$$\begin{aligned}\dot{e}(t)&=A_\sigma e(t)+D_\sigma(f_\sigma(\hat{x},y,u_\sigma)-f_\sigma(x,y,u_\sigma))\\ &\quad-L_\sigma(g_\sigma(\hat{x},u_\sigma)-g_\sigma(x,u_\sigma))-W_\sigma\omega(t).\end{aligned}\tag{3}$$

The following standard assumptions are needed throughout this article.

Assumption 2: *Each subsystem is controllable and observable for any $i\in M$.*

Assumption 3: *σ has finite number of switchings on any finite interval of time.*

Assumption 3 rules out Zeno behaviour for all types of switching (Liberzon 2003).

3. Main results

Define sets

$$\mathscr{H}^i_{q_i,n}=\left\{v^i=(v^i_{11},\dots,v^i_{1n},\dots,v^i_{q_in}):\underline{f}^i_{jk}\le v^i_{jk}\le\bar{f}^i_{jk},\ j=1,\dots,q_i,k=1,\dots,n\right\},\quad\forall i\in M.$$

Each set $\mathscr{H}^i_{q_i,n}$ is a bounded convex domain whose vertices set is

$$\mathscr{V}^i_{q_i,n}=\left\{\alpha^i=(\alpha^i_{11},\dots,\alpha^i_{1n},\dots,\alpha^i_{q_in}):\alpha^i_{jk}\in\{\underline{f}^i_{jk},\bar{f}^i_{jk}\}\right\}.$$

Define the affine matrix functions

$$\mathscr{A}_i(v^i)=A_i+D_i\sum\nolimits_{j,k=1}^{q_i,n}v^i_{jk}e_{q_i}(j)e^T_n(k),\quad v^i\in\mathscr{H}^i_{q_i,n}.\tag{4}$$

By the differential mean value theorem (Zemouche, Boutayeb, and Bara 2008), there exist $z_j(t),\bar{z}_j(t)\in \mathrm{Co}(x(t),\hat{x}(t))$ such that

$$\begin{aligned}&f_i(\hat{x},y,u_i)-f_i(x,y,u_i)\\ &=\left(\sum_{j,k=1}^{q_i,n}e_{q_i}(j)e^T_n(k)\frac{\partial f_{ij}}{\partial x_k}(z_j,y,u_i)\right)e,\end{aligned}\tag{5}$$

$$g_i(\hat{x},u_i)-g_i(x,u_i)=\left(\sum_{j,k=1}^{p_i,n}e_{p_i}(j)e^T_n(k)\frac{\partial g_{ij}}{\partial x_k}(\bar{z}_j,u_i)\right)e.\tag{6}$$

With (4), (5) and

$$h^i(t)=\left(h^i_{11}(t),\dots,h^i_{1n}(t),\dots,h^i_{q_in}(t)\right),$$

$$h^i_{jk}(t)=\frac{\partial f_{ij}}{\partial x_k}(z_j,y,u_i),$$

Equation (3) can be rewritten as

$$\dot{e}(t)=(\mathscr{A}_\sigma(h^\sigma(t))-L_\sigma\mathscr{G}_\sigma(\rho^\sigma(t)))e(t)-W_\sigma\omega(t),\tag{7}$$

where $\mathscr{G}_i(\cdot)$ are given by $\mathscr{G}_i(\rho^i(t)) = \sum_{j,k=1}^{p_i,n} \rho^i_{jk} e_{p_i}(j) e_n^T(k)$, with

$$\rho^i(t) = (\rho^i_{11}(t), \dots, \rho^i_{1n}(t), \dots, \rho^i_{p_in}(t)),$$

$$\rho^i_{jk}(t) = \frac{\partial g_{ij}}{\partial x_k}(\bar{z}_j, u_i).$$

From Assumption 1, $\rho^i(\cdot)$ remains in a bounded domain $\mathscr{F}^i_{p_i,n}$ whose vertices set is

$$\mathscr{W}^i_{p_i,n} = \left\{\beta^i = (\beta^i_{11}, \dots, \beta^i_{1n}, \dots, \beta^i_{p_in}) : \beta^i_{jk} \in \{\underline{g}^i_{jk}, \bar{g}^i_{jk}\}\right\}.$$

In view of $f_i(0, y, u_i) \equiv 0$, there exists $z'_j(t) \in \mathrm{Co}(0, \hat{x})$ such that $D_i f_i(\hat{x}, y, u_i) = (\mathscr{A}_i(h'i(t)) - A_i)\hat{x}$, where $h'^i(t)$ can be defined similarly to $h^i(t)$. Then, the closed-loop system composed of (2), (7) and $u_\sigma = K_\sigma \hat{x}$ is

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}_\sigma \tilde{x}(t) + \tilde{B}_\sigma \omega(t), \\ z(t) &= \tilde{C}_\sigma \tilde{x}(t), \end{aligned} \tag{8}$$

where

$$\tilde{A}_i = \begin{bmatrix} \mathscr{A}_i(h'i(t)) + B_i K_i & -L_i \mathscr{G}_i(\rho^i(t)) \\ 0 & \mathscr{A}_i(h^i(t)) - L_i \mathscr{G}_i(\rho^i(t)) \end{bmatrix},$$

$$\tilde{B}_i = \begin{bmatrix} 0 \\ -W_i \end{bmatrix}, \quad \tilde{C}_i = \begin{bmatrix} E_i & -E_i \\ K_i & 0 \end{bmatrix}, \quad \tilde{x}(t) = \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix}.$$

The H_∞ control problem for the switched system (1) is stated as follows: given a constant $\gamma > 0$, find observer-based dynamic controllers $u_i = K_i \hat{x}$ with (2) for all subsystems and a switching law $i = \sigma(t)$ such that

(i) system (8) is asymptotically stable when $\omega(t) \equiv 0$.
(ii) system (8) has finite L_2-gain γ from the disturbance input $\omega(t)$ to the controlled output $z(t)$, i.e.

$$\int_{t_0}^{\infty} z^T(t) z(t)\,\mathrm{d}t \le \gamma^2 \int_{t_0}^{\infty} \omega^T(t)\omega(t)\,\mathrm{d}t + \upsilon(x(t_0))$$

holds for all $T > 0$, where $x(t_0)$ is the initial state with the initial time t_0 and $\upsilon(\cdot)$ is some real-valued function.

Suppose that we have matrices $P_i > 0$ and symmetric matrices Q_{il} with $Q_{ii} = 0 (i, l \in M)$. Let

$$\begin{aligned} \Omega_i &= \{x \in \Re^n \mid x^T(P_i - P_l + Q_{il})x \le 0, \quad \forall l \in M\}, \\ \tilde{\Omega}_{il} &= \{x \in \Re^n \mid x^T(P_i - P_l + Q_{il})x = 0, l \ne i\}. \end{aligned}$$

Then the switching law is designed as follows:

$$\begin{aligned} &\sigma(t_0) = i, \text{ if } \hat{x}(t_0) \in \Omega_i, \\ &\sigma(t) = \begin{cases} i, & \text{if } \sigma(t^-) = i \text{ and } \hat{x}(t) \in \mathrm{int}\,\Omega_i, \\ l, & \text{if } \sigma(t^-) = i \text{ and } \hat{x}(t) \in \tilde{\Omega}_{il}. \end{cases} \quad t > t_0. \end{aligned} \tag{9}$$

Lemma 1: *For given constants $\eta_{il} \le 0$, $\delta_i > 0$, $\zeta_i > 0$. Suppose that there exist matrices $P_i > 0$, symmetric matrices Q_{il} with $Q_{ii} = 0$, matrix $S > 0$ and matrices R_i such that*

$$\text{Block-diag}\left\{\Psi_i(\alpha^i_1), \Psi_i(\alpha^i_2), \dots, \Psi_i(\alpha^i_{2^{q_in}})\right\} < 0, \tag{10}$$

$$\text{Block-diag}\left\{\Phi_{il}(\alpha^i_1), \Phi_{il}(\alpha^i_2), \dots, \Phi_{il}(\alpha^i_{2^{q_in}})\right\} \le 0, \tag{11}$$

$$\begin{aligned} &\text{Block-diag}\big\{\Gamma_i(\alpha^i_1, \beta^i_1), \dots, \Gamma_i(\alpha^i_{2^{q_in}}, \beta^i_1), \Gamma_i(\alpha^i_1, \beta^i_2), \dots, \\ &\quad \Gamma_i(\alpha^i_{2^{q_in}}, \beta^i_{2^{p_in}})\big\} < 0, \end{aligned} \tag{12}$$

$$Q_{is} + Q_{sl} \le Q_{il}, \tag{13}$$

$$Q_{is} + Q_{sl} \le 0, \tag{14}$$

hold for $\forall\ i,\ s,\ l \in M$, $j = 1, \dots, 2^{q_in}$, $k = 1, \dots, 2^{p_in}$, $\alpha^i_j \in \mathscr{V}^i_{q_i,n}$, $\beta^i_k \in \mathscr{W}^i_{p_i,n}$ where

$$\begin{aligned} \Psi_i(\alpha^i_j) &= \mathscr{A}^T_i(\alpha^i_j)P_i + P_i\mathscr{A}_i(\alpha^i_j) - 2P_iB_iB_i^TP_i + \delta_i I \\ &\quad + \sum_{l \in M, l \ne i} \eta_{il}(P_i - P_l + Q_{il}), \\ \Phi_{il}(\alpha^i_j) &= Q_{il}\big(\mathscr{A}_i(\alpha^i_j) - B_iB_i^TP_i\big) \\ &\quad + \big(\mathscr{A}_i(\alpha^i_j) - B_iB_i^TP_i\big)^T Q_{il}, \\ \Gamma_i(\alpha^i_j, \beta^i_k) &= \mathscr{A}^T_i(\alpha^i_j)S - \mathscr{G}^T_i(\beta^i_k)R_i + S\mathscr{A}_i(\alpha^i_j) \\ &\quad - R_i^T\mathscr{G}_i(\beta^i_k) + \zeta_i I. \end{aligned}$$

Then, the system (8) *with $\omega(t) \equiv 0$ is globally asymptotically stable under the switching law* (9) *and an associated observer-based dynamic controller $u_\sigma = K_\sigma \hat{x}(t)$, the controller and observer gain matrices are $K_i = -B_i^TP_i$ and $L_i = S^{-1}R_i^T$, $i \in M$.*

Proof: We first show how to design an estimation state-dependent switching law, and then achieve asymptotical stability with the help of the GMLFs approach.

Choose the GMLF candidates of the form

$$V(\tilde{x}) = V_{\sigma(t)}(\tilde{x}) = \hat{x}^T P_{\sigma(t)} \hat{x} + \kappa_{\sigma(t)} e^T S e, \tag{15}$$

where $P_i (i \in M)$, S are positive definite matrices satisfying (10)–(12), κ_i are constants to be determined. It is easy to know from (13) and (14) that for any integers $j_1, j_2, \dots, j_q \in M$,

$$Q_{j_1j_2} + Q_{j_2j_3} + \dots + Q_{j_{q-1}j_q} + Q_{j_qj_1} \le 0. \tag{16}$$

Obviously, for each i, the set $\tilde{\Omega}_i = \cup_{l=1, l\ne i}^m \tilde{\Omega}_{il}$ contains the boundary of Ω_i. Moreover, we have $\cup_{i=1}^m \Omega_i = \Re^n \backslash \{0\}$. In fact, if it is false, namely, there exists $\hat{x} \in \Re^n$ satisfying $\hat{x} \notin \Omega_i$, $\forall i$, then we have an integer q and a sequence $j_1, \dots, j_q$, $j_k \ne j_{k+1}$, $k = 1, \dots, q$ with j_{q+1} being considered as j_1 such that

$\hat{x}^T(P_{j_k} - P_{j_{k+1}} + Q_{j_kj_{k+1}})\hat{x} > 0$. Taking the sum over k and noticing (16) yields

$$\sum_{k=1}^{q} \hat{x}^T(P_{j_k} - P_{j_{k+1}} + Q_{j_kj_{k+1}})\hat{x} = \sum_{k=1}^{q} \hat{x}^T Q_{j_kj_{k+1}}\hat{x} \leq 0,$$

which results in contradiction. The sets Ω_i have the property that if $\hat{x} \in \Omega_i \cap \tilde{\Omega}_{il}$ for some i, l and $\hat{x} \in \Re^n$, then $\hat{x} \in \Omega_l$. In fact, $\hat{x} \in \Omega_i \cap \tilde{\Omega}_{il}$ means that $\hat{x}^T(P_i - P_s + Q_{is})\hat{x} \leq 0$ for any $s \in M$ and $\hat{x}^T(P_i - P_l + Q_{il})\hat{x} = 0$. Thus, $P_l = P_i + Q_{\text{il}}$. This in turn gives

$$\hat{x}^T(P_l - P_s + Q_{ls})\hat{x} \leq \hat{x}^T(P_i - P_s + Q_{is})\hat{x} \leq 0.$$

On the other hand, denote the s-th switching instant as t_s. In view of (9) and $\eta_{\text{il}} < 0$ $(i, l \in M)$, we have

$$\eta_{\sigma l}\hat{x}^T \sum_{l\in M, l\neq\sigma} (P_\sigma - P_l + Q_{\sigma l})\hat{x} \geq 0, \ \forall \hat{x} \in \Re^n. \tag{17}$$

With $K_i = -B_i^T P_i$ and $R_i = L_i^T S$, on any interval $[t_s, t_{s+1})$, the time derivative of $V(\tilde{x})$ along the trajectory of the closed-loop system (8) with $\omega(t) = 0$ satisfies

$$\begin{aligned}\dot{V}_\sigma(\tilde{x}) \leq{}& \hat{x}^T\Big(\mathscr{A}_\sigma^T(h'\sigma)P_\sigma + P_\sigma\mathscr{A}_\sigma(h'\sigma) - 2P_{\tau}\sigma P_\sigma\Big)\hat{x} \\ &+ \kappa_\sigma e^T\big(\mathscr{A}_\sigma^T(h^\sigma)S - \mathscr{G}_\sigma^T(\rho^\sigma)R_\sigma + S\mathscr{A}_\sigma(h^\sigma) \\ &- R_\sigma^T\mathscr{G}_\sigma(\rho^\sigma)\big)e \\ &- \hat{x}^T P_\sigma L_\sigma \mathscr{G}_\sigma(\rho^\sigma)e - e^T\mathscr{G}_\sigma^T(\rho^\sigma)L_\sigma P_\sigma\hat{x}. \end{aligned} \tag{18}$$

Denote $F_i(h'i) = \mathscr{A}_i^T(h'i)P_i + P_i\mathscr{A}_i(h'i) - 2P_iB_iB_i^TP_i + \delta_i I$, $\forall\ i \in M$. It is easy to find a common set $\mathscr{V}^i_{q_i,n}$ for h^i and h'^i. Then, combining (17) with (10) yields $F_i(\alpha^i) < 0$ for all $\alpha^i \in \mathscr{V}^i_{q_i,n}$. Using the convexity principle (see Boyd and Vandenberghe (2001) for more details), we deduce that $F_i(h'^i) < 0$ for all h'^i, which means that on any $[t_s, t_{s+1})$,

$$\begin{aligned}&\hat{x}^T\big(\mathscr{A}_\sigma^T(h'\sigma)P_\sigma + P_\sigma\mathscr{A}_\sigma(h'\sigma) - 2P_\sigma B_\sigma B_\sigma^T P_\sigma + \delta_\sigma I\big)\hat{x} \\ &\quad < 0, \quad \forall\hat{x} \neq 0.\end{aligned}$$

Similarly, (12) implies that

$$\begin{aligned}&e^T\big(\mathscr{A}_\sigma^T(h^\sigma)S - \mathscr{G}_\sigma^T(\rho^\sigma)R_\sigma + S\mathscr{A}_\sigma(h^\sigma) - R_\sigma^T\mathscr{G}_\sigma(\rho^\sigma) + \zeta_\sigma I\big)e \\ &\quad < 0\end{aligned}$$

on any $[t_s, t_{s+1})$, $\forall e \neq 0$. Thus, combining these with (18), we have

$$\begin{aligned}\dot{V}(\tilde{x}) &\leq -\delta_\sigma\hat{x}^T\hat{x} - \kappa_\sigma\zeta_\sigma e^T e - \hat{x}^T P_\sigma L_\sigma\mathscr{G}_\sigma(\rho^\sigma)e \\ &\quad - e^T\mathscr{G}_\sigma^T(\rho^\sigma)L_\sigma P_\sigma\hat{x} \\ &= -\tilde{x}^T\Xi_i\tilde{x},\end{aligned}$$

where

$$\Xi_i = \begin{bmatrix} \delta_i I & P_i L_i \mathscr{G}_i(\rho^i) \\ \mathscr{G}_i^T(\rho^i)L_iP_i & \kappa_i\zeta_i I \end{bmatrix}.$$

Choose the positive scalar κ_i large enough such that $\Xi_i > 0$ for all $i \in M$. Therefore, $\dot{V}_{\sigma(t)}(\tilde{x}(t)) < 0$ on any $[t_s, t_{s+1})$.

Also, (11) implies that $Q_{il}(\mathscr{A}_i(h'i(t)) - B_iB_i^TP_i) + (\mathscr{A}_i(h'i(t)) - B_iB_i^TP_i)^TQ_{il} \leq 0$, which tells us that $\hat{x}^T(t)Q_{i_sl}\hat{x}(t)$ are decreasing on $[t_s, t_{s+1})$ along the trajectory of $\dot{\hat{x}}(t) = (\mathscr{A}_i(h'i(t)) + B_iK_i)\hat{x}(t)$. For $s \in N, i_s \in M$, if the i_s-th subsystem is active on $[t_s, t_{s+1})$, according to the switching law (9), at each switching time, we have $\hat{x}_{s+1}^T(P_{i_{s+1}} - P_{i_s} - Q_{i_si_{s+1}})\hat{x}_{s+1} = 0$. For simplicity of notations, suppose $\tilde{x}(t_s) = \tilde{x}_s$. Thus,

$$\begin{aligned}&\hat{x}_{s+1}^T P_{i_{s+1}}\hat{x}_{s+1} - \hat{x}_{s+1}^T P_{i_s}\hat{x}_{s+1} + \hat{x}_{s+2}^T P_{i_{s+2}}\hat{x}_{s+2} \\ &\quad - \hat{x}_{s+2}^T P_{i_{s+1}}\hat{x}_{s+2} \\ &\leq \hat{x}_{s+1}^T Q_{i_si_{s+1}}\hat{x}_{s+1} + \hat{x}_{s+1}^T Q_{i_{s+1}i_{s+2}}\hat{x}_{s+1} \leq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{p=0}^{s}\big(V_{i_{p+1}}(\tilde{x}_{p+1}) - V_{i_p}(\tilde{x}_{p+1})\big) \\ &\quad \leq \begin{cases} 0, & \text{if } s \text{ is odd}, \\ \hat{x}_1^T Q_{i_0i_1}\hat{x}_1 \leq \hat{x}_0^T Q_{i_0i_1}\hat{x}_0, & \text{if } s \text{ is even}. \end{cases}\end{aligned} \tag{19}$$

With this, for any $q \geq 1$, we have

$$\begin{aligned}V_{i_q}(\tilde{x}_q) &= V_{i_0}(\tilde{x}_0) + \sum_{p=1}^{q}\big(V_{i_p}(\tilde{x}_p) - V_{i_{p-1}}(\tilde{x}_p)\big) \\ &\quad + \sum_{p=1}^{q-1}\big(V_{i_p}(\tilde{x}_{p+1}) - V_{i_p}(\tilde{x}_p)\big) \\ &\leq \alpha(\|\tilde{x}_0\|) + \beta(\|\hat{x}_0\|),\end{aligned}$$

where

$$\alpha(r) = \max_{\|\tilde{x}\| \leq r}\{\tilde{x}^T\tilde{P}_i\tilde{x}, i \in M\},$$

$$\beta(r) = \max_{\|\hat{x}\| \leq r}\{|\hat{x}^TQ_{il}\hat{x}|, i, l \in M\}.$$

The GMLFs technique gives the result. $\square$

Remark 1: If scalars η_{il}, δ_i, ζ_i are chosen in advance, conditions in Lemma 1 can be easily transformed into the LMIs:

$$\begin{bmatrix} \begin{array}{c}\mathscr{A}_i^T(\alpha_j^i)P_i + P_i\mathscr{A}_i(\alpha_j^i) + \delta_i I \\ + \sum_{l\in M, l\neq i}\eta_{il}(P_i - P_l + Q_{il})\end{array} & P_iB_i \\ B_i^TP_i & -0.5I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Q_{il}\mathscr{A}_i(\alpha_j^i) + \mathscr{A}_i^T(\alpha_j^i)Q_{il} & Q_{il}B_i & P_iB_i \\ B_i^TQ_{il} & -I & 0 \\ B_i^TP_i & 0 & -I \end{bmatrix} < 0.$$

Remark 2: When $Q_{il} \equiv 0$ $(i, l \in M)$, (11)–(14) are automatically satisfied and (10) becomes the well-known result in Liberzon (2003) and the switching law given by (9) degenerates exactly into the 'min-switching' strategy.

Remark 3: It follows from $f_i(0, y, u_i) \equiv 0$ and (4) that there exists $z'_j \in \mathrm{Co}(0, \hat{x})$ such that $D_i f_i(\hat{x}, y, u_i) = (\mathscr{A}_i(h' i(t)) - A_i)\hat{x}(t)$, where, similar to $h^i(t)$, $h'^i(t)$ can be defined as $h' i(t) = (h' i_{11}(t), \ldots, h' i_{q_i n}(t))$, $h' i_{jk}(t) = \frac{\partial f_{ij}}{\partial x_k}(z'_j, y, u_i)$. It is easy to find a common convex set $\mathscr{H}^i_{q_i,n}$ for $h'^i(t)$ and $h^i(t)$. For instance, let

$$\underline{f}^i_{jk} = \inf\left\{\inf_{Z\in\Re^n\times\Re^{p_i}\times\Re^{m_i}}\left(\frac{\partial f_{ij}}{\partial x_k}(z'_j, y, u_i)\right), \inf_{Z\in\Re^n\times\Re^{p_i}\times\Re^{m_i}}\left(\frac{\partial f_{ij}}{\partial x_k}(z_j, y, u_i)\right)\right\},$$

$$\bar{f}^i_{jk} = \sup\left\{\sup_{Z\in\Re^n\times\Re^{p_i}\times\Re^{m_i}}\left(\frac{\partial f_{ij}}{\partial x_k}(z'_j, y, u_i)\right), \sup_{Z\in\Re^n\times\Re^{p_i}\times\Re^{m_i}}\left(\frac{\partial f_{ij}}{\partial x_k}(z_j, y, u_i)\right)\right\}.$$

Lemma 1 only needs the values of vertices in $\mathscr{V}^i_{q_i,n}$. Therefore, we can suppose that $z'_j(t) = z_j(t)$ without loss of generality.

Remark 4: In GMLF approach, no decreasing requirement of $V_i(\tilde{x}(t))$ on the corresponding active intervals is needed. Lemma 1 only needs $V_i(\tilde{x}(t))$ on an active interval that does not exceed the value of some function of V_i at the 'switched on' instant.

An exponential stabilisability condition can be derived by strengthening the condition of Lemma 2, as shown in the following lemma.

Lemma 2: *The system* (8) *with* $\omega(t) \equiv 0$ *is globally exponentially stable via the switching law* (9) *if the conditions of Lemma* 1 *are satisfied.*

Proof: Let $\varepsilon_i = \underline{\lambda}(\Xi_i)$. Then similar to the proof of Lemma 1, on any $[t_s, t_{s+1})$, we have $\dot{V}(\tilde{x}(t)) < -\varepsilon_\sigma \tilde{x}^T(t)\tilde{x}(t)$. Thus,

$$\begin{aligned}&\dot{V}(\tilde{x}(t)) + \mu_0 V(\tilde{x}(t))\\ &\quad\le -\big(\varepsilon_\sigma - \mu_0\bar{\lambda}(P_\sigma)\big)\|\hat{x}(t)\|^2 - \big(\varepsilon_\sigma - \mu_0\bar{\lambda}(S)\big)\|e(t)\|^2\\ &\quad\le 0\end{aligned}$$

holds for $\mu_0 \le \min_{i\in M}\{\frac{\varepsilon_i}{\bar{\lambda}(P_i)}, \frac{\varepsilon_i}{\lambda\bar{\lambda}(S)}\}$. This implies

$$V(\tilde{x}(t)) \le e^{-\mu_0(t-t_s)}V(\tilde{x}(t_s)),\ t \in [t_s, t_{s+1}).$$

Combining with $P_l = P_i + Q_{il}$ and in view of $\hat{x}^T(t)Q_{i_s l}\hat{x}(t)$ being decreasing on $[t_s, t_{s+1})$, we can easily obtain

$$V(\tilde{x}(t)) \le \begin{cases} e^{-\mu_0(t-t_0)}V(\tilde{x}_0), & \text{if } s \text{ is even},\\ e^{-\mu_0(t-t_0)}V(\tilde{x}_0) + \hat{x}_0^T Q_{i_0 i_1}\hat{x}_0, & \text{if } s \text{ is odd}.\end{cases}$$

Therefore,

$$\|\tilde{x}(t)\|^2 \le \frac{\lambda^+}{\lambda^-}e^{-\mu_0(t-t_0)}\|\tilde{x}(t_0)\|^2,$$

where $\lambda^+ = \max_{i,l\in M}\ \{\bar{\lambda}(P_i) + \bar{\lambda}(Q_{il}),\ \ \bar{\lambda}(S)\}$, $\lambda^- = \min_i\{\underline{\lambda}(P_i), \underline{\lambda}(S)\}$. $\square$

Theorem 1: *Let* $\gamma > 0$ *be a constant. The* H_∞ *control problem for system* (1) *is solved by the switching law* (9) *if the condition of Lemma* 1 *is satisfied with*

$$\begin{aligned}\Psi_i(\alpha^i_j) &= \mathscr{A}_i^T(\alpha^i_j)P_i + P_i\mathscr{A}_i(\alpha^i_j) - P_iB_iB_i^TP_i + 2E_i^TE_i\\ &\quad + \delta_i I + \sum_{l\in M, l\ne i}\eta_{il}(P_i - P_l + Q_{il}),\end{aligned}$$

$$\Gamma_i(\alpha^i_j, \beta^i_k) = \begin{bmatrix}\Upsilon_i(\alpha^i_j, \beta^i_k) & SW_i\\ W_i^TS & -\kappa_i^{-1}\gamma^2 I\end{bmatrix},$$

$$\begin{aligned}\Upsilon_i(\alpha^i_j, \beta^i_k) &= \mathscr{A}_i^T(\alpha^i_j)S - \mathscr{G}_i^T(\beta^i_k)R_i + S\mathscr{A}_i(\alpha^i_j)\\ &\quad - R_i^T\mathscr{G}_i(\beta^i_k) + 2E_i^TE_i + \zeta_i I.\end{aligned}$$

Proof: First, it is easy to see that if the conditions of this theorem are feasible, so are (10)–(12). By Lemma 1, system (8) is asymptotically stabilisable with $\omega(t) \equiv 0$.

Second, introduce $J_T = \int_0^T(z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t))\,\mathrm{d}t$. Applying an argument similar to the proof of Lemma 1 results in

$$\dot{V}(\tilde{x}) + \|z(t)\|^2 - \gamma^2\|\omega(t)\|^2 < 0,\ t \in [t_s, t_{s+1}).$$

Suppose that $t_0 = 0$, when $T \in [t_s, t_{s+1})$, we have

$$\begin{aligned}J_T &= \sum_{p=0}^{s-1}\int_{t_p}^{t_{p+1}}\left(z^Tz - \gamma^2\omega^T\omega + \dot{V}_{i_p}(\tilde{x}(t))\right)\mathrm{d}t\\ &\quad - \sum_{p=0}^{s-1}\big(V_{i_p}(\tilde{x}_{p+1}) - V_{i_p}(\tilde{x}_p)\big) - \big(V_{i_s}(\tilde{x}(T)) - V_{i_s}(\tilde{x}_s)\big)\\ &\quad + \int_{t_p}^{T}\left(z^Tz - \gamma^2\omega^T\omega + \dot{V}_{i_p}(\tilde{x}(t))\right)\mathrm{d}t\\ &\le V_{i_0}(\tilde{x}_0) - V_{i_s}(\tilde{x}(T)) + \sum_{p=0}^{s-1}\big(V_{i_{p+1}}(\tilde{x}_{p+1}) - V_{i_p}(\tilde{x}_{p+1})\big).\end{aligned}$$

Combining this with (19) and the GMLFs technique (Zhao and Hill 2008) leads to

$$J_T \le \upsilon(\tilde{x}_0) := \max_{i_0\in M}\{V_{i_0}(\tilde{x}_0), V_{i_0}(\tilde{x}_0) + \beta(\|\hat{x}_0\|)\}.$$

Let $T \to \infty$, therefore $\int_0^\infty z^T(t)z(t)\,\mathrm{d}t \le \gamma^2\int_0^\infty \omega^T(t) \times \omega(t)\,\mathrm{d}t + \upsilon(\tilde{x}_0)$ holds for all $\omega(t)$. $\square$

4. Example

Consider the switched system of the form (1) with

$$A_1 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & -0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -0.8 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.8 & 1 \\ -1.5 & 2 \\ -0.7 & 2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2 & -0.9 \\ 1 & -1.6 \\ -1 & 0.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.2 \\ -0.3 \\ -0.4 \end{bmatrix},$$

$$W_2 = [-0.5 \quad 0.2 \quad 0.4]^T, \quad E_1 = [1 \quad -1 \quad -2], \\ E_2 = [-0.8 \quad 1 \quad 1],$$

$$f_1 = \begin{bmatrix} 0.2\sin x_1 \\ 3\sin x_3 \end{bmatrix}, \quad g_1 = \begin{bmatrix} x_1 - x_3 \\ x_2 + x_3 \end{bmatrix},$$

$$g_2 = \begin{bmatrix} x_1 + e^{-t}x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}, \quad f_2 = 2\sin x_2, \quad M = \{1, 2\}.$$

Take $\eta_{12} = \eta_{21} = -2$, $\delta_1 = \delta_2 = 0.1$, $\kappa_1 = \kappa_2 = 0.6$, $\zeta_1 = \zeta_2 = 0.5$. Then, the H_∞ control problem of the system (1) is solved by the switching law:

$$\sigma(t) = \begin{cases} 1, & \text{if } \hat{x}(t) \in \Omega_1, \sigma(t^-) = 1 \\ & \quad \text{or } \hat{x}(t) \in \tilde{\Omega}_{21}, \sigma(t^-) = 2, \\ 2, & \text{otherwise} \end{cases}$$

with $\hat{x}(0) = [1, 1, 2]^T$, $\sigma(0) = 2$, $\gamma = 0.8341$ and the controllers with gain matrices

$$K_1 = \begin{bmatrix} -1.9750 & 0.2600 & 2.3355 \\ -1.0248 & -0.0978 & -1.6790 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -2.9173 & -0.8642 & 0.6794 \\ 1.1708 & -0.3173 & -2.8497 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 1.4307 & 0.0905 & -2.7906 \\ -2.3264 & 1.0554 & 1.7820 \end{bmatrix}^T,$$

$$L_2 = \begin{bmatrix} 2.9301 & 0.6167 & 0.5456 \\ 1.2321 & 1.5468 & -0.8335 \end{bmatrix}^T.$$

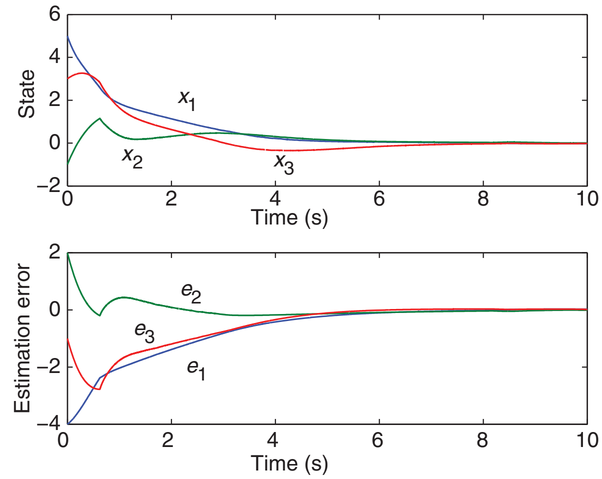

Figure 1. The state response of the system (1) and (3).

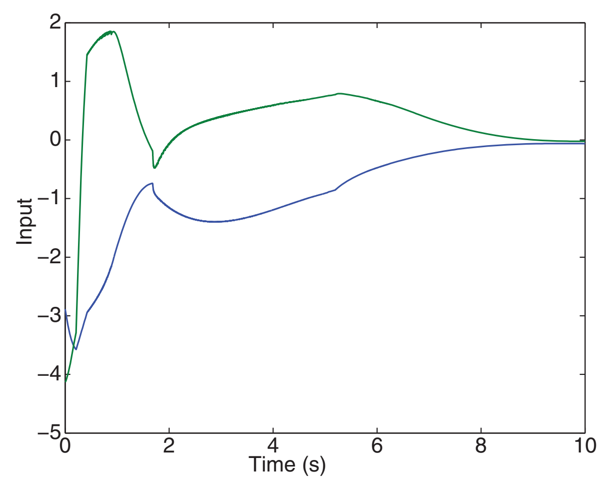

Figure 2. The input signal of the system (1) and (3).

Figure 1 shows the state and estimation error trajectories of the closed-loop system with $x(0) = [5, -1, 3]^T$.

Figure 2 shows the input signal of the switched systems.

5. Conclusion

The problem of the observer-based H_∞ control for a class of switched Lipschitz nonlinear systems has been investigated. By using the GMLFs approach, the controllers, observers and an observer-based switching law, which are independent of the system state, are simultaneously designed. The GMLFs are allowed to be disconnected at the switching times and are allowed to grow on 'switched on' time sequences. This characteristic gives more freedom for the design problem addressed to be solvable as more Lyapunov function candidates for each subsystem are available.

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