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Fault detection for discrete-time LPV systems using interval observers

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ABSTRACT
This paper is concerned with the fault detection (FD) problem for discrete-time linear parameter-varying systems subject to bounded disturbances. A parameter-dependent FD interval observer is designed based on parameter-dependent Lyapunov and slack matrices. The design method is presented by translating the parameter-dependent linear matrix inequalities (LMIs) into finite ones. In contrast to the existing results based on parameter-independent and diagonal Lyapunov matrices, the derived disturbance attenuation, fault sensitivity and nonnegative conditions lead to less conservative LMI characterisations. Furthermore, without the need to design the residual evaluation functions and thresholds, the residual intervals generated by the interval observers are used directly for FD decision. Finally, simulation results are presented for showing the effectiveness and superiority of the proposed method.

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Fault detection; discrete-time linear parameter-varying systems; interval observer; linear matrix inequalities

1. Introduction

Early alarm of faults in engineering systems is a critical issue to improve the performance and reliability. Consequently, fault detection (FD) technique has received much attention in the last decades. Many results have been obtained in the field of model-based FD (see, e.g. Zhong, Ding, Lam, & Wang, 2003, and the references therein). The key technique of model-based FD schemes is to generate the residual signals and thresholds (Frank & Ding, 1997). The remaining important task is to compute a residual evaluation function and compare it with a threshold (Ding, 2008). In the literature, observers and filters have been proven to be the effective residual generators (Emami et al., 2015; Lee & Park, 2015; Li & Yang, 2014; Li & Zhou, 2009). Besides, there are several ways to design the thresholds, including the constant thresholds (Wang & Yang, 2008; Zhong & Yang, 2015) and the time-varying thresholds (Johansson, Bask, & Norlander, 2006; Saijai et al., 2014). There is no denying that threshold setting is especially important and difficult for the dynamic systems subject to exogenous disturbances. Consequently, how to find a new design method of threshold is still an open research problem.

The inherent nonlinear and wide operating range are the non-negligible characteristics in most physical systems. A linear time-invariant model fails to give satisfactory description. As a matter of fact, linear parameter-varying (LPV) technique provides an alternative to represent such practical systems. An LPV system can be regarded as a set of linear systems with different operating points. This idea allows for the ingenious application of the linear system theory for control and estimation problem (Dong & Yang, 2013; Song & Yang, 2011; Zhang, Shi, & Mehr, 2012). On the one hand, compared with a parameter-independent quadratic Lyapunov function, the parameter-dependent one has advantages in reducing the conservatism (Gahinet, Apkarian, & Chilali, 1996; Oliveira & Peres, 2005, 2007). On the other hand, in some cases with the measurable parameters, parameter-dependent controller and filter design is also an effective method to reduce the conservatism (Gao, Lam, & Wang, 2005). The main idea behind this approach is to make full use of the available system information.

Recently, FD for LPV systems has become a very active research area. Many results have been proposed to detect the faults; for example, Grenaille, Henry, and Zolghadri (2008) and Henry (2012) proposed the H_∞/H_- filters to satisfy robustness and fault sensitivity specifications. Armeni, Casavola, and Mosca (2009) presented an FD filter with enhanced fault transmission dc-gains. Hamdi et al. (2012) used a polytopic unknown inputs proportional integral observer to estimate both the states and the faults. However, parameter-independent Lyapunov matrices were used in the above-mentioned FD method. In order to reduce the conservatism, in Rodrigues, Sahnoun, Theilliol, and Ponsart (2013), a polytopic LPV filter was achieved by using the parameter-dependent Lyapunov matrices. Unfortunately, the proposed conditions are still conservative since that the introduced slack matrix is parameter-independent.

Using the concept of interval and zonotope-based algorithm, in De Lira, Puig, Quevedo, and Husar (2011) and De Oca, Puig, and Blesa (2012), the FD observers were designed and the residuals were bounded by the intervals. Furthermore, interval observer has been proven to be very effective in estimating the system states (see, e.g. Cai, Lv, & Zhang, 2012; Efimov, Perruquetti, Raïssi, & Zolghadri, 2013; Efimov, Raïssi, & Zolghadri, 2013; Mazenc & Bernard, 2011; Mazenc, Dinh, & Niculescu, 2014). The main idea is to guarantee the nonnegativity or cooperativity of the error dynamics. Furthermore, Chebotarev, Efimov, Raïssi, and Zolghadri (2015) provided the interval estimations for continuous-time LPV systems. The performance optimisation problem of the interval observer was considered in the L_1/L_2 sense. But the

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interval observer design conditions for the generic case were expressed as linear matrix inequalities (LMIs) under the assumption that the Lyapunov matrix is diagonal. Such a relatively conservative method has been used to detect the faults for T–S fuzzy systems in Rotondo et al. (2016). This encourages us to investigate a more effective method for designing the interval observers without the additional restrictions.

Based on the above considerations, this paper presents an interval observer-based FD method for a class of discrete-time LPV systems. First, a parameter-dependent FD interval observer is designed by taking into account the bounds of the disturbances. Second, l_1 and H_∞ performances are introduced to enhance the robustness of the residual intervals and the sensitivity to faults, respectively. Then, the disturbance attenuation, fault sensitivity and nonnegative conditions are translated into finite LMIs. Note that it is the first time to characterise the disturbance attenuation, fault sensitivity and nonnegative conditions for interval observers simultaneously. Furthermore, the FD decision is made by determining whether the zero value is excluded from the residual interval when the faults occur. To the author's knowledge, parameter-dependent interval observer-based FD scheme for LPV systems has not been fully investigated. The main contributions from this perspective are

- Compared with the traditional observer-based FD method in Frank and Ding (1997), Grenaille et al. (2008), Wang and Yang (2008, 2009), Zhong and Yang (2015) and Zhong et al. (2003), an interval observer is designed such that the generated residual intervals can be used directly to make the FD decision. It avoids the design of residual evaluation functions and threshold generators;
- Compared with the existing interval observer results in Chebotarev et al. (2015) and Rotondo et al. (2016), in this paper, the dilated LMI technique is used to remove the additional restrictions that the Lyapunov matrices are diagonal;
- Compared with the existing parameter-independent Lyapunov functions-based FD schemes in Armeni et al. (2009), Hamdi et al. (2012) and parameter-independent slack matrices-based FD scheme in Rodrigues et al. (2013), parameter-dependent Lyapunov and slack matrices are applied in this paper such that the obtained observer design conditions are less conservative.

The rest of the paper is organised as follows. The system description and problem statement are presented in Section 2. Section 3 provides the main results for designing the FD interval observer. Simulation examples are given in Section 4 followed by some conclusions.

Notations: The notation $\|\cdot\|$ denotes the Euclidean norm for vectors and $|\cdot|$ denotes the absolute value for scalars. l_∞ and l_2 norms of the signal $z(k)$ are defined as $\|z(k)\|_\infty = ess\,\sup_{t\geq 0}\|z(k)\|$ and $\|z(k)\|_2 = (\Sigma_{k=0}^{\infty}\|z(k)\|^2)^{\frac{1}{2}}$, respectively. The space of signals l_∞ is defined to be $l_\infty = \{\|z(k)\|_\infty < \infty\}$ and the space of signal l_2 is defined to be $l_2 = \{\|z(k)\|_2 < \infty\}$. For vectors $\Phi = [\Phi_i]_{n\times 1}$, $\Psi = [\Psi_i]_{n\times 1}$, we define $\Phi \preceq \Psi$ ($\Phi \succeq \Psi$) by $\Phi_i \leq \Psi_i$ ($\Phi_i \geq \Psi_i$), $\forall 1 \leq i \leq n$. $Q > 0$ ($Q < 0$) means the matrix Q is positive (negative) definite. For given matrix $M \in \mathbb{R}^{m\times n}$, define $M^+ = \max\{0, M\}$ and $M^- = M^+ - M$. A matrix $N \in \mathbb{R}^{m\times n}$ is said to be nonnegative (positive) if all its elements are nonnegative (positive). A system $\xi(k+1) = M\xi(k) + Nd(k)$ with $\xi(k) \in \mathbb{R}^n$, $d(k) \in \mathbb{R}^{n_d}$ is called nonnegative if $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n_d}$ are nonnegative, $d(k) \succeq 0$ and $\xi(0) \succeq 0$.

2. System description and problem statement

2.1. System description

Consider the following discrete-time LPV system:

$$\begin{aligned} x(k+1) &= A(\alpha)x(k) + B(\alpha)\omega(k) + E(\alpha)f(k) \\ y(k) &= C(\alpha)x(k), \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^{n_y}$ are the state and the measured output. $\omega(k) \in \mathbb{R}^{n_d}$ denotes the disturbance and $f(k) \in \mathbb{R}^{n_f}$ is the fault that belongs to l_2. The matrices $A(\alpha)$, $B(\alpha)$, $E(\alpha)$ and $C(\alpha)$ are the known constant matrices. It is assumed that $\Omega(\alpha) = (A(\alpha), B(\alpha), E(\alpha), C(\alpha)) \in \mathscr{R}$, where $\mathscr{R}$ is a given convex bounded polyhedral domain described by n vertices:

$$\mathscr{R} = \left\{\Omega(\alpha) = \sum_{i=1}^{N}\rho_i(\alpha)\Omega_i;\ \sum_{i=1}^{N}\rho_i(\alpha) = 1,\ \rho_i(\alpha) \geq 0\right\}$$

and $\Omega_i = (A_i, B_i, E_i, C_i)$ denotes the ith vertex of the polytope. It is also assumed that α is measured online and does not depend explicitly on the time variable, what is more, (C_i, A_i) is observable.

In order to generate residuals, the interval observer is designed in this paper. The following assumptions and lemmas are required for the design and analysis.

Assumption 2.1: *There exist the known bound functions $\underline{\omega}(k) \in \mathbb{R}^{n_d}$ and $\overline{\omega}(k) \in \mathbb{R}^{n_d}$ such that*

$$\underline{\omega}(k) \preceq \omega(k) \preceq \overline{\omega}(k). \quad (2)$$

Remark 2.1: Assumption 2.1 means that the upper and lower bounds on the unknown disturbances are required, which is common in the interval observer literature (Chebotarev et al., 2015; Efimov et al., 2013; Mazenc & Bernard, 2011; Mazenc et al., 2014; Rotondo et al., 2016) and the l_1 filter literature (Wang & Yang, 2014).

Lemma 2.1 (Wang & Yang, 2014)**:** *The following conditions are equivalent:*

(1) *There exists a symmetric matrix $P > 0$ such that*

$$A^TPA + Q < 0.$$

(2) *There exists a symmetric matrix $P > 0$ and a matrix G such that*

$$\begin{bmatrix} Q & -A^TG \\ -G^TA & P - G - G^T \end{bmatrix} < 0.$$

2.2. Problem statement

The task of this paper is to detect the faults by designing a parameter-dependent interval observer. First, the interval observer is constructed as follows:

$$\begin{aligned}\underline{x}(k+1) &= (A(\alpha)-\underline{L}(\alpha)C(\alpha))\underline{x}(k)+\underline{L}(\alpha)y(k)+B^{+}(\alpha)\underline{\omega}(k)\\ &\quad -B^{-}(\alpha)\overline{\omega}(k)-\underline{F}(\alpha)(\overline{x}(k)-\underline{x}(k))\\ \overline{x}(k+1) &= (A(\alpha)-\overline{L}(\alpha)C(\alpha))\overline{x}(k)+\overline{L}(\alpha)y(k)+B^{+}(\alpha)\overline{\omega}(k)\\ &\quad -B^{-}(\alpha)\underline{\omega}(k)+\overline{F}(\alpha)(\overline{x}(k)-\underline{x}(k))\\ \underline{y}(k) &= C^{+}(\alpha)\underline{x}(k)-C^{-}(\alpha)\overline{x}(k)\\ \overline{y}(k) &= C^{+}(\alpha)\overline{x}(k)-C^{-}(\alpha)\underline{x}(k)\\ \underline{r}(k) &= V(\alpha)(y(k)-\overline{y}(k))\\ \overline{r}(k) &= V(\alpha)(y(k)-\underline{y}(k)),\end{aligned}\tag{3}$$

where $\underline{x}(k)\in\mathbb{R}^n$ and $\overline{x}(k)\in\mathbb{R}^n$ denote the lower and upper estimates of the state $x(k)$, respectively. $\underline{y}(k)\in\mathbb{R}^{n_y}$ and $\overline{y}(k)\in\mathbb{R}^{n_y}$ are the lower and upper estimates of the output $y(k)$. $\underline{r}(k)\in\mathbb{R}^{n_f}$ and $\overline{r}(k)\in\mathbb{R}^{n_f}$ are the lower and upper residuals. $B^{+}(\alpha)=\sum_{i=1}^{N}\rho_i(\alpha)B_i^{+}$, $B^{-}(\alpha)=\sum_{i=1}^{N}\rho_i(\alpha)B_i^{-}$, $C^{+}(\alpha)=\sum_{i=1}^{N}\rho_i(\alpha)C_i^{+}$ and $C^{-}(\alpha)=\sum_{i=1}^{N}\rho_i(\alpha)C_i^{-}$. $\underline{L}(\alpha)\in\mathbb{R}^{n\times n_y}$, $\overline{L}(\alpha)\in\mathbb{R}^{n\times n_y}$, $\underline{F}(\alpha)\in\mathbb{R}^{n\times n}$ and $\overline{F}(\alpha)\in\mathbb{R}^{n\times n}$ are the parameter-dependent observer gain matrices and $V(\alpha)\in\mathbb{R}^{n_f\times n_y}$ is a positive matrix to be determined.

Define $\underline{e}(k)=x(k)-\underline{x}(k)$, $\overline{e}(k)=\overline{x}(k)-x(k)$, then the error dynamics is given by

$$\begin{aligned}\underline{e}(k+1) &= (A(\alpha)-\underline{L}(\alpha)C(\alpha)+\underline{F}(\alpha))\underline{e}(k)+\underline{F}(\alpha)\overline{e}(k)\\ &\quad +B(\alpha)\omega(k)-(B^{+}(\alpha)\underline{\omega}(k)-B^{-}(\alpha)\overline{\omega}(k))\\ &\quad +E(\alpha)f(k)\\ \overline{e}(k+1) &= (A(\alpha)-\overline{L}(\alpha)C(\alpha)+\overline{F}(\alpha))\overline{e}(k)+\overline{F}(\alpha)\underline{e}(k)\\ &\quad +(B^{+}(\alpha)\overline{\omega}(k)-B^{-}(\alpha)\underline{\omega}(k))-B(\alpha)\omega(k)\\ &\quad -E(\alpha)f(k)\end{aligned}\tag{4}$$

Furthermore, denoting $\xi(k)=\begin{bmatrix}\underline{e}(k)\\ \overline{e}(k)\end{bmatrix}$, $r(k)=\begin{bmatrix}\underline{r}(k)\\ \overline{r}(k)\end{bmatrix}$ and $\tilde{\omega}(k)=\begin{bmatrix}\omega(k)-\underline{\omega}(k)\\ \overline{\omega}(k)-\omega(k)\end{bmatrix}$, we get the following augmented system:

$$\begin{aligned}\xi(k+1) &= \tilde{A}(\alpha)\xi(k)+\tilde{B}(\alpha)\tilde{\omega}(k)+\tilde{E}(\alpha)f(k)\\ r(k) &= \tilde{C}(\alpha)\xi(k),\end{aligned}\tag{5}$$

where $\tilde{A}(\alpha)=\begin{bmatrix}A(\alpha)-\underline{L}(\alpha)C(\alpha)+\underline{F}(\alpha) & \underline{F}(\alpha)\\ \overline{F}(\alpha) & A(\alpha)-\overline{L}(\alpha)C(\alpha)+\overline{F}(\alpha)\end{bmatrix}$, $\tilde{B}(\alpha)=\begin{bmatrix}B^{+}(\alpha) & B^{-}(\alpha)\\ B^{-}(\alpha) & B^{+}(\alpha)\end{bmatrix}$, $\tilde{E}(\alpha)=\begin{bmatrix}E(\alpha)\\ -E(\alpha)\end{bmatrix}$ and $\tilde{C}(\alpha)=\begin{bmatrix}-V(\alpha)C^{-}(\alpha) & -V(\alpha)C^{+}(\alpha)\\ V(\alpha)C^{+}(\alpha) & V(\alpha)C^{-}(\alpha)\end{bmatrix}$.

It is clear that $\tilde{B}(\alpha)\succeq 0$ and $\tilde{\omega}(k)\succeq 0$ hold. Inspired by the idea of nonnegative system, we have $\xi(k)\succeq 0$ hold in the fault-free case if the matrices $A(\alpha)-\underline{L}(\alpha)C(\alpha)+\underline{F}(\alpha)$, $A(\alpha)-\overline{L}(\alpha)C(\alpha)+\overline{F}(\alpha)$, $\underline{F}(\alpha)$, $\overline{F}(\alpha)$ are nonnegative and $\xi(0)\succeq 0$. Furthermore, the following relations can be obtained:

- $\underline{e}(k)\succeq 0$ and $\overline{e}(k)\succeq 0$;
- $\underline{x}(k)\preceq x(k)\preceq\overline{x}(k)$;
- $\underline{y}(k)\preceq y(k)\preceq\overline{y}(k)$;
- $0\in[\underline{r}_m(k),\overline{r}_m(k)]$ for $m=1,2,\ldots,n_f$.

The FD problem is formulated as follows: determine the FD interval observer (3) such that

(i) $\tilde{A}(\alpha)$ is Schur stable and nonnegative,

(ii) (l_1 performance). The l_1 norm of the operator from $\tilde{\omega}(k)$ to $r(k)$ is less than β under zero initial conditions

$$\sup_{\tilde{\omega}(k)\in l_\infty}\frac{\|r(k)\|_\infty}{\|\tilde{\omega}(k)\|_\infty}<\beta,\tag{6}$$

(iii) (H_∞ performance). The l_2 norm of the operator from $f(k)$ to $r(k)-Jf(k)$ is less than γ under zero initial conditions

$$\|r(k)-Jf(k)\|_2<\gamma\|f(k)\|_2\tag{7}$$

for all $f(k)\in l_2$, where $J_1\in\mathbb{R}^{n_f\times n_f}$ is a weighting matrix, $J=\left[J_1^T\ J_1^T\right]^T$.

Specification (i) is the condition for the nonnegativity and the stability of the error system. Specification (ii) represents the robustness of the upper and lower residuals against amplitude-bounded disturbances in the l_1 norm sense, while the H_∞ performance given in specification (iii) is used to improve the sensitivity of the upper and lower residuals to faults indirectly.

Remark 2.2: It should be noted that in (7), the weighting matrix J is introduced to transform the fault sensitivity specification into an H_∞ constraint. The effects of $f(k)$ on $r(k)$ are maximised indirectly. Similar methods have been used in Li and Yang (2014) and the references therein.

Remark 2.3: Compared with the traditional FD methods for LPV systems, e.g. H_∞ filter-based method in Wang and Yang (2009), Luenberger observer-based method in Wei and Verhaegen (2011), the great advantage of the interval observer-based FD method is that (3) can generate not only the residual signals, but also the thresholds.

3. Main results

3.1. Disturbance attenuation condition

In this subsection, a parameter-dependent Lyapunov function is considered and the disturbance attenuation condition is given in the following theorem.

Theorem 3.1: *Given a positive scalar β, the augmented system (5) is stable and satisfies the l_1 performance (6) if there exist matrices $P_1(\alpha)=\begin{bmatrix}P_{11}(\alpha) & P_{12}(\alpha)\\ P_{12}^T(\alpha) & P_{13}(\alpha)\end{bmatrix}>0$, $W(\alpha)=\begin{bmatrix}W_{11}(\alpha) & 0\\ 0 & W_{22}(\alpha)\end{bmatrix}$, $X(\alpha)$, $Y(\alpha)$, $R(\alpha)$, $S(\alpha)$, a positive matrix $V(\alpha)$ and positive scalars μ, $0<\lambda<1$ such that the following inequalities hold:*

$$\Phi(\alpha)<0,\tag{8}$$

$$\Omega(\alpha)<0,\tag{9}$$

where

$$\Phi(\alpha)=\begin{bmatrix} -\lambda P_{11}(\alpha) & -\lambda P_{12}(\alpha) & 0 & 0 & -A^T(\alpha)W_{11}(\alpha)+C^T(\alpha)X(\alpha)-R(\alpha) & -S(\alpha)\\ * & -\lambda P_{13}(\alpha) & 0 & 0 & -R(\alpha) & -A^T(\alpha)W_{22}(\alpha)+C^T(\alpha)Y(\alpha)-S(\alpha)\\ * & * & -\mu I & 0 & -B^{+T}(\alpha)W_{11}(\alpha) & -B^{-T}(\alpha)W_{22}(\alpha)\\ * & * & * & -\mu I & -B^{-T}(\alpha)W_{11}(\alpha) & -B^{+T}(\alpha)W_{22}(\alpha)\\ * & * & * & * & P_{11}(\alpha)-W_{11}(\alpha)-W_{11}^T(\alpha) & P_{12}(\alpha)\\ * & * & * & * & * & P_{13}(\alpha)-W_{22}(\alpha)-W_{22}^T(\alpha) \end{bmatrix},$$

$$\Omega(\alpha)=\begin{bmatrix} -(1-\lambda)P_{11}(\alpha) & -(1-\lambda)P_{12}(\alpha) & 0 & -C^{-T}(\alpha)V^T(\alpha) & C^{+T}(\alpha)V^T(\alpha)\\ * & -(1-\lambda)P_{13}(\alpha) & 0 & -C^{+T}(\alpha)V^T(\alpha) & C^{-T}(\alpha)V^T(\alpha)\\ * & * & -(\beta-\mu)I & 0 & 0\\ * & * & * & -\beta I & 0\\ * & * & * & * & -\beta I \end{bmatrix}.$$

Proof: Considering the parameter-dependent Lyapunov function $V_1(\xi(k))=\xi^T(k)P_1(\alpha)\xi(k)$, system (5) with $f(k)=0$ satisfies the l_1 performance (6) if the following inequalities hold:

$$V_1\ (\xi(k+1))-\lambda V_1(\xi(k))-\mu\tilde{\omega}^T(k)\tilde{\omega}(k)<0, \quad (10)$$

$$r^T(k)\ r(k)-\beta[(1-\lambda)V_1(\xi(k))+(\beta-\mu)\tilde{\omega}^T(k)\tilde{\omega}(k)]<0. \quad (11)$$

From

$$\begin{aligned} &V_1(\xi(k+1))-\lambda V_1(\xi(k))-\mu\tilde{\omega}^T(k)\tilde{\omega}(k)\\ &=\xi^T(\tilde{A}^T(\alpha)P_1(\alpha)\tilde{A}(\alpha)-\lambda P_1(\alpha))\xi\\ &\quad+2\xi^T\tilde{A}^T(\alpha)P_1(\alpha)\tilde{B}(\alpha)\tilde{\omega}\\ &\quad+\tilde{\omega}^T\tilde{B}^T(\alpha)P_1(\alpha)\tilde{B}(\alpha)\tilde{\omega}-\mu\tilde{\omega}^T\tilde{\omega}\\ &=[\xi^T\ \tilde{\omega}^T]\\ &\quad\times\begin{bmatrix}\tilde{A}^T(\alpha)P_1(\alpha)\tilde{A}(\alpha)-\lambda P_1(\alpha) & \tilde{A}^T(\alpha)P_1(\alpha)\tilde{B}(\alpha)\\ * & \tilde{B}^T(\alpha)P_1(\alpha)\tilde{B}(\alpha)-\mu I\end{bmatrix}\\ &\quad\times\begin{bmatrix}\xi\\ \tilde{\omega}\end{bmatrix}, \end{aligned}$$

we have (10) is equivalent to

$$\begin{bmatrix}\tilde{A}^T(\alpha)\\ \tilde{B}^T(\alpha)\end{bmatrix}P_1(\alpha)\left[\tilde{A}(\alpha)\ \tilde{B}(\alpha)\right]+\begin{bmatrix}-\lambda P_1(\alpha) & 0\\ * & -\mu I\end{bmatrix}<0.$$

Applying Lemma 2.1, we introduce the parameter-dependent slack variable $W(\alpha)$, then the above inequality is equivalent to

$$\begin{bmatrix}-\lambda P_1(\alpha) & 0 & -\tilde{A}^T(\alpha)W(\alpha)\\ * & -\mu I & -\tilde{B}^T(\alpha)W(\alpha)\\ * & * & P_1(\alpha)-W(\alpha)-W^T(\alpha)\end{bmatrix}<0.$$

Denoting $X(\alpha)=\underline{L}^T(\alpha)W_{11}(\alpha)$, $Y(\alpha)=\overline{L}^T(\alpha)W_{22}(\alpha)$, $R(\alpha)=\underline{F}^T(\alpha)W_{11}(\alpha)$ and $S(\alpha)=\overline{F}^T(\alpha)W_{22}(\alpha)$, we have

$$\begin{bmatrix}\Phi_{11}(\alpha) & 0 & \Phi_{13}(\alpha)\\ * & -\mu I & \Phi_{23}(\alpha)\\ * & * & \Phi_{33}(\alpha)\end{bmatrix}<0, \quad (12)$$

where

$$\Phi_{11}(\alpha)=\begin{bmatrix}-\lambda P_{11}(\alpha) & -\lambda P_{12}(\alpha)\\ * & -\lambda P_{13}(\alpha)\end{bmatrix},$$

$$\Phi_{13}(\alpha)=\begin{bmatrix}-A^T(\alpha)W_{11}(\alpha)+C^T(\alpha)X(\alpha)-R(\alpha) & -S(\alpha)\\ -R(\alpha) & -A^T(\alpha)W_{22}(\alpha)+C^T(\alpha)Y(\alpha)-S(\alpha)\end{bmatrix},$$

$$\Phi_{23}(\alpha)=-\begin{bmatrix}B^{+T}(\alpha)W_{11}(\alpha) & B^{-T}(\alpha)W_{22}(\alpha)\\ B^{-T}(\alpha)W_{11}(\alpha) & B^{+T}(\alpha)W_{22}(\alpha)\end{bmatrix},$$

$$\Phi_{33}(\alpha)=\begin{bmatrix}P_{11}(\alpha)-W_{11}(\alpha)-W_{11}^T(\alpha) & P_{12}(\alpha)\\ * & P_{13}(\alpha)-W_{22}(\alpha)-W_{22}^T(\alpha)\end{bmatrix}.$$

On the other hand, from

$$\begin{aligned} &\beta^{-1}r^T(k)r(k)-[(1-\lambda)V_1(\xi(k))+(\beta-\mu)\tilde{d}^T(k)\tilde{d}(k)]\\ &=\beta^{-1}\xi^T\tilde{C}^T(\alpha)\tilde{C}(\alpha)\xi-(1-\lambda)\xi^TP_1(\alpha)\xi-(\beta-\mu)\tilde{\omega}^T\tilde{\omega}\\ &=[\xi^T\ \tilde{\omega}^T]\begin{bmatrix}\beta^{-1}\tilde{C}^T(\alpha)\tilde{C}(\alpha)-(1-\lambda)P_1(\alpha) & 0\\ * & -(\beta-\mu)I\end{bmatrix}\\ &\quad\times\begin{bmatrix}\xi\\ \tilde{\omega}\end{bmatrix}, \end{aligned}$$

we have the following inequality implies (11)

$$\begin{bmatrix}\Omega_{11}(\alpha) & 0 & \Omega_{13}(\alpha)\\ * & -(\beta-\mu)I & 0\\ * & * & -\beta I\end{bmatrix}<0, \quad (13)$$

where

$$\Omega_{11}(\alpha)=\begin{bmatrix}-(1-\lambda)P_{11}(\alpha) & -(1-\lambda)P_{12}(\alpha)\\ * & -(1-\lambda)P_{13}(\alpha)\end{bmatrix},$$

$$\Omega_{13}(\alpha)=\begin{bmatrix}-C^{-T}(\alpha)V^T(\alpha) & C^{+T}(\alpha)V^T(\alpha)\\ -C^{+T}(\alpha)V^T(\alpha) & C^{-T}(\alpha)V^T(\alpha)\end{bmatrix}.$$

It is easy to see that (8) implies (12) and (9) implies (13). This implies that the specification (ii) is satisfied if the inequality conditions (8) and (9) are feasible. ■

3.2. Fault sensitivity condition

The following theorem presents the fault sensitivity condition for the interval observer.

Theorem 3.2: *Given a positive scalar γ, the augmented system (5) satisfies the H_∞ performance (7) if there exist matrices $P_2(\alpha) = \begin{bmatrix} P_{21}(\alpha) & P_{22}(\alpha) \\ P_{22}^T(\alpha) & P_{23}(\alpha) \end{bmatrix} > 0$, $W(\alpha) = \begin{bmatrix} W_{11}(\alpha) & 0 \\ 0 & W_{22}(\alpha) \end{bmatrix}$, $X(\alpha)$, $Y(\alpha)$, $R(\alpha)$, $S(\alpha)$, J and a positive matrix $V(\alpha)$ such that the following inequality holds:*

$$\Psi(\alpha) < 0, \tag{14}$$

where

$$\Psi(\alpha) = \begin{bmatrix} -P_{21}(\alpha) & -P_{22}(\alpha) & 0 & -C^{-T}(\alpha)V^T(\alpha) & C^{+T}(\alpha)V^T(\alpha) & \begin{matrix} -A^T(\alpha)W_{11}(\alpha) \\ +C^T(\alpha)X(\alpha) - R(\alpha) \end{matrix} & -S(\alpha) \\ * & -P_{23}(\alpha) & 0 & -C^{+T}(\alpha)V^T(\alpha) & C^{-T}(\alpha)V^T(\alpha) & -R(\alpha) & \begin{matrix} -A^T(\alpha)W_{22}(\alpha) \\ +C^T(\alpha)Y(\alpha) - S(\alpha) \end{matrix} \\ * & * & -\gamma^2 I & -J_1^T & -J_1^T & -E^T(\alpha)W_{11}(\alpha) & E^T(\alpha)W_{22}(\alpha) \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & P_{21}(\alpha) - W_{11}(\alpha) - W_{11}^T(\alpha) & P_{22}(\alpha) \\ * & * & * & * & * & * & P_{23}(\alpha) - W_{22}(\alpha) - W_{22}^T(\alpha) \end{bmatrix}.$$

Proof: Considering the parameter-dependent Lyapunov function $V_2(\xi(k)) = \xi^T(k)P_2(\alpha)\xi(k)$, system (5) with $\tilde{\omega}(k) = 0$ satisfies the H_∞ performance (7) if the following inequality holds:

$$V_2(\xi(k+1)) - V_2(\xi(k)) + (r(k) - Jf(k))^T(r(k) - Jf(k)) - \gamma^2 f^T(k)f(k) < 0. \tag{15}$$

From

$$\begin{aligned} &\Delta V_2 + (r - Jf)^T(r - Jf) - \gamma^2 f^T f \\ &= \xi^T(\tilde{A}^T(\alpha)P_2(\alpha)\tilde{A}(\alpha) - P_2(\alpha))\xi \\ &\quad + 2\xi^T\tilde{A}^T(\alpha)P_2(\alpha)\tilde{E}(\alpha)f + f^T\tilde{E}^T(\alpha)P_2(\alpha)\tilde{E}(\alpha)f \\ &\quad + \xi^T\tilde{C}^T(\alpha)\tilde{C}(\alpha)\xi - 2\xi^T\tilde{C}^T(\alpha)Jf + f^TJ^TJf - \gamma^2 f^T f \\ &= \begin{bmatrix} \xi^T & f^T \end{bmatrix} \begin{bmatrix} \tilde{A}^T(\alpha)P_2(\alpha)\tilde{A}(\alpha) - P_2(\alpha) + \tilde{C}^T(\alpha)\tilde{C}(\alpha) & \tilde{A}^T(\alpha)P_2(\alpha)\tilde{E}(\alpha) - \tilde{C}^T(\alpha)J \\ * & \tilde{E}^T(\alpha)P_2(\alpha)\tilde{E}(\alpha) + J^TJ - \gamma^2 I \end{bmatrix} \begin{bmatrix} \xi \\ f \end{bmatrix}, \end{aligned}$$

we have (15) is equivalent to

$$\begin{bmatrix} \tilde{A}^T(\alpha)P_2(\alpha)\tilde{A}(\alpha) - P_2(\alpha) + \tilde{C}^T(\alpha)\tilde{C}(\alpha) & \tilde{A}^T(\alpha)P_2(\alpha)\tilde{E}(\alpha) - \tilde{C}^T(\alpha)J \\ * & \tilde{E}^T(\alpha)P_2(\alpha)\tilde{E}(\alpha) + J^TJ - \gamma^2 I \end{bmatrix} < 0.$$

By using Schur complements, the above inequality can be rewritten as

$$\begin{bmatrix} \tilde{A}^T(\alpha) \\ \tilde{E}^T(\alpha) \\ 0 \end{bmatrix} P_2(\alpha) \begin{bmatrix} \tilde{A}(\alpha) & \tilde{E}(\alpha) & 0 \end{bmatrix} + \begin{bmatrix} -P_2(\alpha) & 0 & \tilde{C}^T(\alpha) \\ * & -\gamma^2 I & -J^T \\ * & * & -I \end{bmatrix} < 0. \tag{16}$$

The same slack variable $W(\alpha)$ is introduced by applying Lemma 2.1, then the following inequality implies (16)

$$\begin{bmatrix} -P_2(\alpha) & 0 & \tilde{C}^T(\alpha) & -\tilde{A}^T(\alpha)W(\alpha) \\ * & -\gamma^2 I & -J^T & -\tilde{E}^T(\alpha)W(\alpha) \\ * & * & -I & 0 \\ * & * & * & P_2(\alpha) - W(\alpha) - W^T(\alpha) \end{bmatrix} < 0.$$

Similar to Theorem 3.1, denoting $X(\alpha) = \underline{L}^T(\alpha)W_{11}(\alpha)$, $Y(\alpha) = \overline{L}^T(\alpha)W_{22}(\alpha)$, $R(\alpha) = \underline{F}^T(\alpha)W_{11}(\alpha)$ and $S(\alpha) = \overline{F}^T(\alpha)W_{22}(\alpha)$, one gets

$$\begin{bmatrix} \Psi_{11}(\alpha) & 0 & \tilde{C}^T(\alpha) & \Psi_{14}(\alpha) \\ * & -\gamma^2 I & -J^T & \Psi_{24}(\alpha) \\ * & * & -I & 0 \\ * & * & * & \Psi_{44}(\alpha) \end{bmatrix} < 0, \tag{17}$$

where

$$\Psi_{11}(\alpha) = \begin{bmatrix} -P_{21}(\alpha) & -P_{22}(\alpha) \\ * & -P_{23}(\alpha) \end{bmatrix},$$

$$\Psi_{14}(\alpha) = \begin{bmatrix} -A^T(\alpha)W_{11}(\alpha) + C^T(\alpha)X(\alpha) - R(\alpha) & -S(\alpha) \\ -R(\alpha) & -A^T(\alpha)W_{22}(\alpha) + C^T(\alpha)Y(\alpha) - S(\alpha) \end{bmatrix},$$

$$\Psi_{24}(\alpha) = \begin{bmatrix} -E^T(\alpha)W_{11}(\alpha) & E^T(\alpha)W_{22}(\alpha) \end{bmatrix},$$

$$\Psi_{44}(\alpha) = \begin{bmatrix} P_{21}(\alpha) - W_{11}(\alpha) - W_{11}^T(\alpha) & P_{22}(\alpha) \\ * & P_{23}(\alpha) - W_{22}(\alpha) - W_{22}^T(\alpha) \end{bmatrix}.$$

It can be seen that (14) implies (17). This implies that the specification (iii) is satisfied if the inequality condition (14) is feasible. ■

3.3. Nonnegative condition

In this subsection, the nonnegative condition of the matrix $\tilde{A}(\alpha)$ is expressed as parameter-dependent LMIs. Owing to the dilated LMI approach in Lemma 2.1, the Lyapunov matrices are decoupled from the system matrices. Based on this technique, the nonnegative restriction can be transferred from the Lyapunov matrices to the slack matrix variables. The following theorem is given for an nth-order system.

Theorem 3.3: *$\tilde{A}(\alpha)$ is nonnegative if there exist positive matrices*

$$W_{11}(\alpha)=\begin{bmatrix} W_{111}(\alpha) & 0 & \cdots & 0\\ 0 & W_{112}(\alpha) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & W_{11n}(\alpha)\end{bmatrix},$$

$$W_{22}(\alpha)=\begin{bmatrix} W_{221}(\alpha) & 0 & \cdots & 0\\ 0 & W_{222}(\alpha) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & W_{22n}(\alpha)\end{bmatrix},$$

$$R(\alpha)=\begin{bmatrix} R_{11}(\alpha) & R_{12}(\alpha) & \cdots & R_{1n}(\alpha)\\ R_{21}(\alpha) & R_{22}(\alpha) & \cdots & R_{2n}(\alpha)\\ \vdots & \vdots & \ddots & \vdots\\ R_{n1}(\alpha) & R_{n2}(\alpha) & \cdots & R_{nn}(\alpha)\end{bmatrix},$$

$$S(\alpha)=\begin{bmatrix} S_{11}(\alpha) & S_{12}(\alpha) & \cdots & S_{1n}(\alpha)\\ S_{21}(\alpha) & S_{22}(\alpha) & \cdots & S_{2n}(\alpha)\\ \vdots & \vdots & \ddots & \vdots\\ S_{n1}(\alpha) & S_{n2}(\alpha) & \cdots & S_{nn}(\alpha)\end{bmatrix},$$

and matrices

$$X(\alpha)=\begin{bmatrix}X_1(\alpha) & X_2(\alpha) & \cdots & X_n(\alpha)\end{bmatrix},\quad Y(\alpha)=\begin{bmatrix}Y_1(\alpha) & Y_2(\alpha) & \cdots & Y_n(\alpha)\end{bmatrix}$$

such that the following parameter-dependent LMIs hold:

$$\Pi_{n1}\geq 0,\ \Pi_{n2}\geq 0, \tag{18}$$

where

$$\Pi_{n1}=W_{11g}(\alpha)a_{gh}(\alpha)-X_g^T(\alpha)c_h(\alpha)+R_{hg}(\alpha),$$
$$\Pi_{n2}=W_{22g}(\alpha)a_{gh}(\alpha)-Y_g^T(\alpha)c_h(\alpha)+S_{hg}(\alpha),$$
$$g,h=1,2,\ldots,n.$$

Proof: First, consider a second-order system

$$A(\alpha)=(a_{gh}(\alpha))_{2\times 2},\ C(\alpha)=\begin{bmatrix}c_1(\alpha) & c_2(\alpha)\end{bmatrix},$$

and the interval observer gain matrices

$$\underline{L}(\alpha)=\begin{bmatrix}\underline{l}_1(\alpha)\\ \underline{l}_2(\alpha)\end{bmatrix},\quad \overline{L}(\alpha)=\begin{bmatrix}\overline{l}_1(\alpha)\\ \overline{l}_2(\alpha)\end{bmatrix},$$
$$\underline{F}(\alpha)=\begin{bmatrix}\underline{f}_{11}(\alpha) & \underline{f}_{12}(\alpha)\\ \underline{f}_{21}(\alpha) & \underline{f}_{22}(\alpha)\end{bmatrix},\quad \overline{F}(\alpha)=\begin{bmatrix}\overline{f}_{11}(\alpha) & \overline{f}_{12}(\alpha)\\ \overline{f}_{21}(\alpha) & \overline{f}_{22}(\alpha)\end{bmatrix},$$

then,

$$A(\alpha)-\underline{L}(\alpha)C(\alpha)+\underline{F}(\alpha)$$
$$=\begin{bmatrix} a_{11}(\alpha)-\underline{l}_1^T(\alpha)c_1(\alpha)+\underline{f}_{11}(\alpha) & a_{12}(\alpha)-\underline{l}_1^T(\alpha)c_2(\alpha)+\underline{f}_{12}(\alpha)\\ a_{21}(\alpha)-\underline{l}_2^T(\alpha)c_1(\alpha)+\underline{f}_{21}(\alpha) & a_{22}(\alpha)-\underline{l}_2^T(\alpha)c_2(\alpha)+\underline{f}_{22}(\alpha)\end{bmatrix},$$
$$A(\alpha)-\overline{L}(\alpha)C(\alpha)+\overline{F}(\alpha)$$
$$=\begin{bmatrix} a_{11}(\alpha)-\overline{l}_1^T(\alpha)c_1(\alpha)+\overline{f}_{11}(\alpha) & a_{12}(\alpha)-\overline{l}_1^T(\alpha)c_2(\alpha)+\overline{f}_{12}(\alpha)\\ a_{21}(\alpha)-\overline{l}_2^T(\alpha)c_1(\alpha)+\overline{f}_{21}(\alpha) & a_{22}(\alpha)-\overline{l}_2^T(\alpha)c_2(\alpha)+\overline{f}_{22}(\alpha)\end{bmatrix}. \tag{19}$$

Based on the definition of the nonnegative matrix, the nonnegative restriction is equivalent to

$$a_{gh}(\alpha)-\underline{l}_g^T(\alpha)c_h(\alpha)+\underline{f}_{gh}(\alpha)\geq 0,\ a_{gh}(\alpha)-\overline{l}_g^T(\alpha)c_h(\alpha)+\overline{f}_{gh}(\alpha)\geq 0,\ g,h=1,2. \tag{20}$$

Theorems 3.1 and 3.2 imply that $A(\alpha)-\underline{L}(\alpha)C(\alpha)+\underline{F}(\alpha)$ and $A(\alpha)-\overline{L}(\alpha)C(\alpha)+\overline{F}(\alpha)$ are coupled with the slack matrices $W_{11}(\alpha)$ and $W_{22}(\alpha)$, respectively. Choosing $W_{11}(\alpha)=\begin{bmatrix}W_{111}(\alpha) & 0\\ 0 & W_{112}(\alpha)\end{bmatrix}>0$, $W_{22}(\alpha)=\begin{bmatrix}W_{221}(\alpha) & 0\\ 0 & W_{222}(\alpha)\end{bmatrix}>0$ and denoting

$$X^T(\alpha)=W_{11}^T(\alpha)\underline{L}(\alpha)=\begin{bmatrix}X_1^T(\alpha)\\ X_2^T(\alpha)\end{bmatrix},$$
$$Y^T(\alpha)=W_{22}^T(\alpha)\overline{L}(\alpha)=\begin{bmatrix}Y_1^T(\alpha)\\ Y_2^T(\alpha)\end{bmatrix},$$
$$R^T(\alpha)=W_{11}^T(\alpha)\underline{F}(\alpha)=\begin{bmatrix}R_{11}(\alpha) & R_{21}(\alpha)\\ R_{12}(\alpha) & R_{22}(\alpha)\end{bmatrix},$$
$$S^T(\alpha)=W_{12}^T(\alpha)\overline{F}(\alpha)=\begin{bmatrix}S_{11}(\alpha) & S_{21}(\alpha)\\ S_{12}(\alpha) & S_{22}(\alpha)\end{bmatrix},$$

left-multiplying (19) by $W_{11}^T(\alpha)$ and $W_{22}^T(\alpha)$, we have the nonnegative conditions (20) can be converted into the following equivalent inequalities:

$$\Pi_{21}\geq 0,\ \Pi_{22}\geq 0,$$

where

$$\Pi_{21}=W_{11g}(\alpha)a_{gh}(\alpha)-X_g^T(\alpha)c_h(\alpha)+R_{hg}(\alpha),$$
$$\Pi_{22}=W_{22g}(\alpha)a_{gh}(\alpha)-Y_g^T(\alpha)c_h(\alpha)+S_{hg}(\alpha),$$
$$g,h=1,2.$$

Similarly, for an nth-order system

$$A(\alpha)=(a_{gh}(\alpha))_{n\times n},\ C(\alpha)=\begin{bmatrix}c_1(\alpha) & c_2(\alpha) & \cdots & c_n(\alpha)\end{bmatrix},$$

and the interval observer gain matrices

$$\underline{L}(\alpha)=\begin{bmatrix}\underline{l}_1^T(\alpha) & \underline{l}_2^T(\alpha) & \cdots & \underline{l}_n^T(\alpha)\end{bmatrix}^T,$$
$$\overline{L}(\alpha)=\begin{bmatrix}\overline{l}_1^T(\alpha) & \overline{l}_2^T(\alpha) & \cdots & \overline{l}_n^T(\alpha)\end{bmatrix}^T,$$

$$\underline{F}(\alpha) = \begin{bmatrix} \underline{f}_{11}(\alpha) & \underline{f}_{12}(\alpha) & \cdots & \underline{f}_{1n}(\alpha) \\ \underline{f}_{21}(\alpha) & \underline{f}_{22}(\alpha) & \cdots & \underline{f}_{2n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ \underline{f}_{n1}(\alpha) & \underline{f}_{n2}(\alpha) & \cdots & \underline{f}_{nn}(\alpha) \end{bmatrix},$$

$$\overline{F}(\alpha) = \begin{bmatrix} \overline{f}_{11}(\alpha) & \overline{f}_{12}(\alpha) & \cdots & \overline{f}_{1n}(\alpha) \\ \overline{f}_{21}(\alpha) & \overline{f}_{22}(\alpha) & \cdots & \overline{f}_{2n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{f}_{n1}(\alpha) & \overline{f}_{n2}(\alpha) & \cdots & \overline{f}_{nn}(\alpha) \end{bmatrix},$$

we can obtain the nonnegative parameter-dependent LMI conditions (18) for the nth-order system. ■

Remark 3.1: Since the parameter-dependent slack matrix $W(\alpha)$ is introduced here, the Lyapunov matrices $P_1(\alpha)$ and $P_2(\alpha)$ in Theorems 3.1 and 3.2 are more general matrices without any structural restrictions.

3.4. LMI conditions for the parameter-dependent interval observer design

In this subsection, the above disturbance attenuation, fault sensitivity and nonnegative conditions will be translated into the LMIs using the extended Polya's theorem in Oliveira and Peres (2005). The following definitions are needed before presenting the main results. Define $\mathscr{K}(d)$ as the set of N-tuples obtained as all possible combinations of $k_1k_2\cdots k_N$, $k_i \in \mathbb{Z}_+$, $i = 1, 2, \ldots, N$ such that $k_1 + k_2 + \cdots + k_N = d$. $\mathscr{K}_l(d)$ is the lth N-tuple of $\mathscr{K}(d)$ which is lexically ordered, $l = 1, 2, \ldots, J(d)$. For a fixed N, the number of elements in $\mathscr{K}(d)$ is given $J(d) = \frac{(N+d-1)!}{d!(N-1)!}$ and the associated standard multinomial coefficients are $\mathscr{X}^l(d) = \frac{d!}{(k_1!k_2!\cdots k_N!)}$, $k_1k_2\cdots k_N = \mathscr{K}_l(d)$, $l = 1, 2, \ldots, J(d)$. Consider the following multinomial coefficients:

$$\mathscr{X}_i^l(d, a) = \begin{cases} \dfrac{d!}{k_1!\cdots(k_i-a)!\cdots k_N!}, & \text{if } k_i - a \in \mathbb{Z}_+; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathscr{X}_{ij}^l(d, a, b) = \begin{cases} \dfrac{d!}{k_1!\cdots(k_i-a)!\cdots(k_j-b)!\cdots k_N!}, & \text{if } k_i - a \in \mathbb{Z}_+, \; k_j - b \in \mathbb{Z}_+; \\ 0, & \text{otherwise.} \end{cases}$$

all of them depending on $k_1k_2\cdots k_N = \mathscr{K}_l(d)$, $l = 1, 2, \ldots, J(d)$.

Theorem 3.4: *The conditions in (8), (9), (14) and (18) hold if and only if there exists a sufficiently large d, matrices* $P_1(\alpha) = \sum_{i=1}^N \rho_i(\alpha)P_{1i} = \sum_{i=1}^N \rho_i(\alpha)\begin{bmatrix} P_{11i} & P_{12i} \\ P_{12i}^T & P_{13i}\end{bmatrix} > 0$, $P_2(\alpha) = \sum_{i=1}^N \rho_i(\alpha)P_{2i} = \sum_{i=1}^N \rho_i(\alpha)\begin{bmatrix} P_{21i} & P_{22i} \\ P_{22i}^T & P_{23i}\end{bmatrix} > 0$, $J = \begin{bmatrix} J_1 \\ J_1 \end{bmatrix}$, $X(\alpha) = \sum_{i=1}^N \rho_i(\alpha)X_i$, $Y(\alpha) = \sum_{i=1}^N \rho_i(\alpha)Y_i$ *with* $X_i = [X_{1i}\ X_{2i}\ \cdots\ X_{ni}]$, $Y_i = [Y_{1i}\ Y_{2i}\ \cdots\ Y_{ni}]$ *and positive matrices*

$$V(\alpha) = \sum_{i=1}^N \rho_i(\alpha)V_i,$$

$$R(\alpha) = \sum_{i=1}^N \rho_i(\alpha)R_i = \sum_{i=1}^N \rho_i(\alpha)\begin{bmatrix} R_{11i} & R_{12i} & \cdots & R_{1ni} \\ R_{21i} & R_{22i} & \cdots & R_{2ni} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1i} & R_{n2i} & \cdots & R_{nni} \end{bmatrix},$$

$$S(\alpha) = \sum_{i=1}^N \rho_i(\alpha)S_i = \sum_{i=1}^N \rho_i(\alpha)\begin{bmatrix} S_{11i} & S_{12i} & \cdots & S_{1ni} \\ S_{21i} & S_{22i} & \cdots & S_{2ni} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1i} & S_{n2i} & \cdots & S_{nni} \end{bmatrix},$$

$$W(\alpha) = \sum_{i=1}^N \rho_i(\alpha)W_i = \sum_{i=1}^N \rho_i(\alpha)\begin{bmatrix} W_{11i} & 0 \\ 0 & W_{22i} \end{bmatrix}$$

with

$$W_{11i} = \begin{bmatrix} W_{111i} & 0 & \cdots & 0 \\ 0 & W_{112i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{11ni} \end{bmatrix},$$

$$W_{22i} = \begin{bmatrix} W_{221i} & 0 & \cdots & 0 \\ 0 & W_{222i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{22ni} \end{bmatrix}$$

such that for $l = 1, 2, \ldots, J(d+2)$, *the following LMIs hold:*

$$\Phi_l < 0,\ \Omega_l < 0,\ \Psi_l < 0,\ \Pi_{n1l} \geq 0,\ \Pi_{n2l} \geq 0, \tag{21}$$

where

$$\Phi_l = \sum_{i=1}^N \mathscr{X}_i^l(d,2)\Phi_{ii} + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \mathscr{X}_{ij}^l(d,1,1)(\Phi_{ij} + \Phi_{ji})$$

$$\Omega_l = \sum_{i=1}^N \mathscr{X}_i^l(d,2)\Omega_{ii} + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \mathscr{X}_{ij}^l(d,1,1)(\Omega_{ij} + \Omega_{ji})$$

$$\Psi_l = \sum_{i=1}^N \mathscr{X}_i^l(d,2)\Psi_{ii} + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \mathscr{X}_{ij}^l(d,1,1)(\Psi_{ij} + \Psi_{ji})$$

$$\Pi_{n1l} = \sum_{i=1}^N \mathscr{X}_i^l(d,2)\Pi_{n1ii} + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \mathscr{X}_{ij}^l(d,1,1)(\Pi_{n1ij} + \Pi_{n1ji})$$

$$\Pi_{n2l} = \sum_{i=1}^{N} \mathscr{X}_i^l(d,2)\Pi_{n2ii} + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \mathscr{X}_{ij}^l(d,1,1)(\Pi_{n2ij} + \Pi_{n2ji})$$

$k_1k_2\cdots k_N = \mathscr{K}_l(d+2)$ *and*

$$\Phi_{ij} = \begin{bmatrix} -\lambda P_{11i} & -\lambda P_{12i} & 0 & 0 & -A_j^T W_{11i} + C_j^T X_i - R_i & -S_i \\ * & -\lambda P_{13i} & 0 & 0 & -R_i & -A_j^T W_{22i} + C_j^T Y_i - S_i \\ * & * & -\mu I & 0 & -B_j^{+T} W_{11i} & -B_j^{-T} W_{22i} \\ * & * & * & -\mu I & -B_j^{-T} W_{11i} & -B_j^{+T} W_{22i} \\ * & * & * & * & P_{11i} - W_{11i} - W_{11i}^T & P_{12i} \\ * & * & * & * & * & P_{13i} - W_{22i} - W_{22i}^T \end{bmatrix},$$

$$\Omega_{ij} = \begin{bmatrix} -\lambda P_{11i} & -\lambda P_{12i} & 0 & -C_j^{-T} V_i^T & C_j^{+T} V_i^T \\ * & -\lambda P_{13i} & 0 & -C_j^{+T} V_i^T & C_j^{-T} V_i^T \\ * & * & -(\beta-\mu) I & 0 & 0 \\ * & * & * & -\beta I & 0 \\ * & * & * & * & -\beta I \end{bmatrix},$$

$$\Psi_{ij} = \begin{bmatrix} -P_{21i} & -P_{22i} & 0 & -C_j^{-T} V_i^T & C_j^{+T} V_i^T & -A_j^T W_{11i} + C_j^T X_i - R_i & -S_i \\ * & -P_{23i} & 0 & -C_j^{+T} V_i^T & C_j^{-T} V_i^T & -R_i & -A_j^T W_{22i} + C_j^T Y_i - S_i \\ * & * & -\gamma^2 I & -J_1^T & -J_1^T & -E_j^T W_{11i} & E_j^T W_{22i} \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & P_{21i} - W_{11i} - W_{11i}^T & P_{22i} \\ * & * & * & * & * & * & P_{23i} - W_{22i} - W_{22i}^T \end{bmatrix},$$

$$\Pi_{n1ij} = W_{11gi} a_{ghj} - X_{gi}^T c_{hj} + R_{hgi},$$
$$\Pi_{n2ij} = W_{22gi} a_{ghj} - Y_{gi}^T c_{hj} + S_{hgi},$$
$$g, h = 1, 2, \ldots, n.$$

Proof: The proof is immediate by combining Theorems 3.1, 3.2, 3.3 and Theorem 5 in Oliveira and Peres (2005). ■

Remark 3.2: By setting $P_1(\alpha) = P_2(\alpha) = W(\alpha)$, Theorem 3.4 can be reduced to the case that the Lyapunov matrices are diagonal. The interval observer design with diagonal Lyapunov matrices has been considered in Chebotarev et al. (2015) and Rotondo et al. (2016). There is no doubt that the results in Chebotarev et al. (2015) and Rotondo et al. (2016) are conservative. It can be seen from Examples 1 and 2 that the FD observer design given in Theorem 3.4 is less conservative than the reduced results based on diagonal Lyapunov matrices.

In order to compare with the parameter-independent results, Theorem 3.4 can be reduced to the following corollary.

Corollary 3.1: *The parameter-independent interval observer* (3) *can be designed if there exist positive scalars* β, μ, $0 < \lambda < 1$, γ, *matrices* $P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{13} \end{bmatrix} > 0$, $P_2 = \begin{bmatrix} P_{21} & P_{22} \\ P_{22}^T & P_{23} \end{bmatrix} > 0$, $J = \begin{bmatrix} J_1 \\ J_1 \end{bmatrix}$, $X = [X_1\ X_2\ \cdots\ X_n]$, $Y = [Y_1\ Y_2\ \cdots\ Y_n]$, *and positive matrices* V,

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix},$$
$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$
$$W = \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix}$$

with

$$W_{11} = \begin{bmatrix} W_{111} & 0 & \cdots & 0 \\ 0 & W_{112} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{11n} \end{bmatrix}$$

and

$$W_{22} = \begin{bmatrix} W_{221} & 0 & \cdots & 0 \\ 0 & W_{222} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{22n} \end{bmatrix}$$

such that for $i = 1, 2, \ldots, N$, *the following LMIs hold:*

$$\Phi_i < 0,\ \Omega_i < 0,\ \Psi_i < 0,\ \Pi_{n1i} \geq 0,\ \Pi_{n2i} \geq 0, \qquad (22)$$

where

$$\Phi_i = \begin{bmatrix} -\lambda P_{11} & -\lambda P_{12} & 0 & 0 & -A_i^T W_{11} + C_i^T X - R & -S \\ * & -\lambda P_{13} & 0 & 0 & -R & -A_i^T W_{22} + C_i^T Y - S \\ * & * & -\mu I & 0 & -B_i^{+T} W_{11} & -B_i^{-T} W_{22} \\ * & * & * & -\mu I & -B_i^{-T} W_{11} & -B_i^{+T} W_{22} \\ * & * & * & * & P_{11} - W_{11} - W_{11}^T & P_{12} \\ * & * & * & * & * & P_{13} - W_{22} - W_{22}^T \end{bmatrix},$$

$$\Omega_i = \begin{bmatrix} -\lambda P_{11} & -\lambda P_{12} & 0 & -C_i^{-T}V^T & C_i^{+T}V^T \\ * & -\lambda P_{13} & 0 & -C_i^{+T}V^T & C_i^{-T}V^T \\ * & * & -(\beta-\mu)I & 0 & 0 \\ * & * & * & -\beta I & 0 \\ * & * & * & * & -\beta I \end{bmatrix},$$

$$\Psi_i = \begin{bmatrix} -P_{21} & -P_{22} & 0 & -C_i^{-T}V^T & C_i^{+T}V^T & -A_i^TW_{11}+C_i^TX-R & -S \\ * & -P_{23} & 0 & -C_i^{+T}V^T & C_i^{-T}V^T & -R & -A_i^TW_{22}+C_i^TY-S \\ * & * & -\gamma^2 I & -J_1^T & -J_1^T & -E_i^TW_{11} & E_i^TW_{22} \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & P_{21}-W_{11}-W_{11}^T & P_{22} \\ * & * & * & * & * & * & P_{23}-W_{22}-W_{22}^T \end{bmatrix},$$

$$\Pi_{n1i} = W_{11g}a_{ghi} - X_g^T c_{hi} + R_{hg},$$

$$\Pi_{n2i} = W_{22g}a_{ghi} - Y_g^T c_{hi} + S_{hg}.$$

then, the parameter-independent interval observer gain matrices $\underline{L}$, $\overline{L}$, $\underline{F}$ *and* $\overline{F}$ *can be determined as follows:* $\underline{L} = W_{11}^{-T}X^T$, $\overline{L} = W_{22}^{-T}Y^T$, $\underline{F} = W_{11}^{-T}R^T$ *and* $\overline{F} = W_{22}^{-T}S^T$.

Remark 3.3: The parameter-independent Lyapunov functions and slack matrices are applied and the parameter-independent observer gain matrices are obtained in Corollary 3.1. It is obvious that Corollary 3.1 is conservative because the FD interval observer design does not make full use of the available system information. It will be proven in Examples 1 and 2 that the result in Theorem 3.4 is less conservative than the result in Corollary 3.1.

Based on Theorems 3.1, 3.2, 3.3 and 3.4, the parameter-dependent interval observer is designed by solving the following convex optimisation problem:

$$\begin{aligned} \min \quad & \varepsilon_1\beta + \varepsilon_2\gamma \\ s.t.(21) \quad & for\ all \quad l = 1, 2, \ldots, J(d+2), \end{aligned} \tag{23}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are the weighting factors. Then, the observer gain matrices can be constructed as

$$\underline{L}(\alpha) = \left(\sum_{i=1}^{N} \rho_i(\alpha)W_{11i}^T\right)^{-1} \sum_{i=1}^{N} \rho_i(\alpha)X_i^T,$$

$$\overline{L}(\alpha) = \left(\sum_{i=1}^{N} \rho_i(\alpha)W_{22i}^T\right)^{-1} \sum_{i=1}^{N} \rho_i(\alpha)Y_i^T,$$

$$\underline{F}(\alpha) = \left(\sum_{i=1}^{N} \rho_i(\alpha)W_{11i}^T\right)^{-1} \sum_{i=1}^{N} \rho_i(\alpha)R_i^T,$$

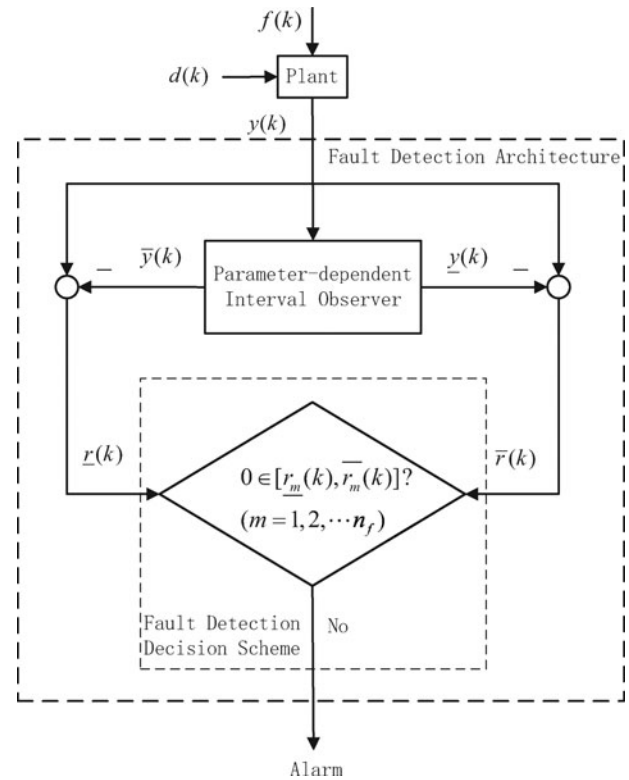

Figure 1. Architecture of the FD scheme.

$$\overline{F}(\alpha) = \left(\sum_{i=1}^{N} \rho_i(\alpha)W_{22i}^T\right)^{-1} \sum_{i=1}^{N} \rho_i(\alpha)S_i^T.$$

3.5. *Fault detection decision scheme*

In this subsection, we combine the upper and lower residuals to detect the faults. Considering the properties of the interval observer and the specification (ii), the relations $0 \in [\underline{r}_m(k), \overline{r}_m(k)]$ hold for all $m = 1, 2, \ldots, n_f$ and the robustness of the residual interval constructed by $\underline{r}_m(k)$ and $\overline{r}_m(k)$ is increased in the fault-free case. When the faults occur, the nonnegativity of the dynamics (4) will not be guaranteed. Furthermore, the specification (iii) maximises the effects of the faults on the upper and lower residuals.

The detailed design procedure is illustrated in Figure 1 and the corresponding FD decision scheme is made as follows:

Fault detection decision scheme. When at least one component of the upper and lower residuals satisfies $0 \notin [\underline{r}_m(k), \overline{r}_m(k)]$, then alarm.

Remark 3.4: The aforementioned FD decision is made by determining whether the zero value is excluded from the residual intervals $[\underline{r}_m(k), \overline{r}_m(k)]$ when the faults occur. Moreover, the lower residual $\underline{r}_m(k)$ and the upper residual $\overline{r}_m(k)$ are directly generated by (3). Compared with the classical FD methods in Ding (2008), Frank and Ding (1997), Grenaille et al. (2008), Wang and Yang (2008, 2009), Zhong and Yang (2015) and Zhong et al. (2003), the advantage of the proposed interval observer-based FD decision scheme lies in the fact that it avoids the design of residual evaluation functions and thresholds.

4. Simulation examples

4.1. Example 1

To demonstrate the effectiveness and the advantages of the proposed FD method, the benchmark mass-spring systems borrowed from Lim and How (2002) is considered.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -f \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} f(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

where k and f are the stiffness and friction coefficients, respectively. We are interested in considering the varying parameter $k = k_0(1+\alpha)$, where $k_0 = 1$, $f = 1$ and α is a measurable parameter satisfying $|\alpha| \le 1$. Using the zero-order hold equivalent method, with a sampling period $T = 0.1$ s, a discrete-time model of the system is given by

$$x(k+1) = Ax(k) + Bd(k) + Ef(k)$$
$$y(k) = Cx(k),$$

where $A = \begin{bmatrix} 1 & T \\ -T(1+\alpha) & 1-T \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ T \end{bmatrix}$, $E = \begin{bmatrix} T \\ T \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The system can be modelled by a two-vertex polytope with

$$\rho_1(\alpha) = \frac{1+\alpha}{2}, \ \rho_2(\alpha) = \frac{1-\alpha}{2}.$$

When $d = 3$, fixing $\lambda = 0.9$ and solving the optimisation problem (23) with the weighting factors $\varepsilon_1 = \varepsilon_2 = 1$, we get the following Lyapunov matrices, slack matrices and matrix variables:

$$P_{11} = \begin{bmatrix} 21.4582 & -1.0529 & 0.0295 & -0.1101 \\ -1.0529 & 1.6143 & -0.1101 & -0.0497 \\ 0.0295 & -0.1101 & 21.4582 & -1.0529 \\ -0.1101 & -0.0497 & -1.0529 & 1.6143 \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} 21.4577 & -1.0539 & 0.0299 & -0.1091 \\ -1.0539 & 1.6119 & -0.1091 & -0.0472 \\ 0.0299 & -0.1091 & 21.4577 & -1.0539 \\ -0.1091 & -0.0472 & -1.0539 & 1.6119 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 11.2916 & -0.8835 & 2.0880 & 0.3948 \\ -0.8835 & 1.5490 & 0.3948 & 0.0079 \\ 2.0880 & 0.3948 & 11.2916 & -0.8835 \\ 0.3948 & 0.0079 & -0.8835 & 1.5490 \end{bmatrix},$$

$$P_{22} = \begin{bmatrix} 9.0514 & -0.7163 & 1.5543 & 0.3918 \\ -0.7163 & 1.5046 & 0.3918 & -0.0196 \\ 1.5543 & 0.3918 & 9.0514 & -0.7163 \\ 0.3918 & -0.0196 & -0.7163 & 1.5046 \end{bmatrix},$$

$$X_1 = Y_1 = \begin{bmatrix} 12.2653 & -0.0271 \end{bmatrix},$$
$$X_2 = Y_2 = \begin{bmatrix} 12.2654 & 0.2620 \end{bmatrix},$$

$$R_1 = R_2 = S_1 = S_2 = \begin{bmatrix} 0.9802 & 0.2620 \\ 0.0000 & 0.0000 \end{bmatrix},$$

$$W_{111} = W_{221} = W_{112} = W_{222} = \begin{bmatrix} 11.2851 & 0 \\ 0 & 1.4455 \end{bmatrix},$$

$$V_1 = V_2 = 1.0000.$$

Table 1. Different cases for weighting factors.

ρ	$\rho\ \varepsilon_1 = 10, \varepsilon_2 = 1$	$\varepsilon_1 = 1, \varepsilon_2 = 1$	$\varepsilon_1 = 1, \varepsilon_2 = 10$
β	0.3410	0.4849	1.2772
γ	0.8050	0.6058	0.3508

Table 2. Performance comparison.

	β	γ
Theorem 3.4	0.4849	0.6058
Remark 3.2	0.8944	0.9457
Corollary 3.1	0.5679	0.6590

Without loss of generality, assume $\alpha = 0.5$, then $\rho_1(\alpha) = 0.75$, $\rho_2(\alpha) = 0.25$, the interval observer gain matrices can be given as follows:

$$\underline{L}(\alpha) = \left(\sum_{i=1}^{2} \rho_i(\alpha) W_{11i}^T\right)^{-1} \sum_{i=1}^{2} \rho_i(\alpha) X_i^T = \begin{bmatrix} 1.0869 & 0.0312 \end{bmatrix}^T,$$

$$\overline{L}(\alpha) = \left(\sum_{i=1}^{2} \rho_i(\alpha) W_{22i}^T\right)^{-1} \sum_{i=1}^{2} \rho_i(\alpha) Y_i^T = \begin{bmatrix} 1.0869 & 0.0312 \end{bmatrix}^T,$$

$$\underline{F}(\alpha) = \left(\sum_{i=1}^{2} \rho_i(\alpha) W_{11i}^T\right)^{-1} \sum_{i=1}^{2} \rho_i(\alpha) R_i^T = \begin{bmatrix} 0.0869 & 0.0000 \\ 0.1812 & 0.0000 \end{bmatrix},$$

$$\overline{F}(\alpha) = \left(\sum_{i=1}^{2} \rho_i(\alpha) W_{22i}^T\right)^{-1} \sum_{i=1}^{2} \rho_i(\alpha) S_i^T = \begin{bmatrix} 0.0869 & 0.0000 \\ 0.1812 & 0.0000 \end{bmatrix}.$$

Assume that the initial conditions are $\underline{x}(0) = x(0) = \overline{x}(0) = 0$, the disturbance is $d(k) = 0.2 + 0.1|\cos(0.05k)|$. The known upper and lower bounds are $\overline{d}(k) = 0.3$ and $\underline{d}(k) = 0.2$. The fault signal is set up as

$$f(k) = \begin{cases} 0.2, & k \ge 100; \\ 0, & \text{otherwise}. \end{cases}$$

The results are shown in Figures 2 and 3. Figure 2 shows the trajectories of system output $y(k)$, lower output estimation $\underline{y}(k)$ and upper output estimation $\overline{y}(k)$, From Figure 2(a), it is clear that the relation $\underline{y}(k) \preceq y(k) \preceq \overline{y}(k)$ holds in the fault-free case, while in Figure 2(b), the relation is broken after a fault occurs. The trajectories of the lower residual $\underline{r}(k)$ and the upper residual $\overline{r}(k)$ are shown in Figure 3. From Figure 3(a), it is also obvious that the relation $0 \in [\underline{r}(k), \overline{r}(k)]$ always holds in the fault-free case. At the same time, $0 \in [\underline{r}(k), \overline{r}(k)]$ holds before $k = 100$ in Figure 3(b) and $0 \notin [\underline{r}(k), \overline{r}(k)]$ after $k = 101$, then the fault can be detected.

Besides, Table 1 illustrates that there is a trade-off between the disturbance attenuation and the fault sensitivity performances.

As mentioned in Remark 3.2, the values of disturbance attenuation and fault sensitivity levels can be obtained by setting $P_{11} = P_{21} = W_1$, $P_{12} = P_{22} = W_2$, $\varepsilon_1 = \varepsilon_2 = 1$, $V_1 = V_2 = 1.0000$. Table 2 presents a comparison result. Moreover, the values of β and γ derived from Corollary 3.1 are also shown in Table 2. It illustrates that the proposed design method achieves

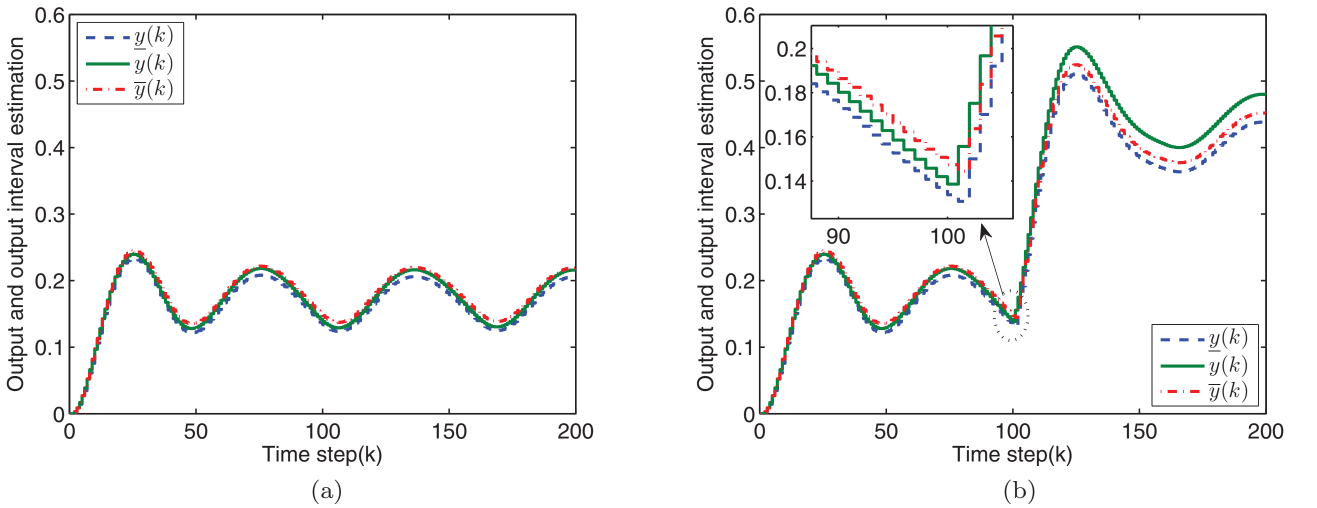

Figure 2. Outputs and output interval estimations in both the fault-free (a) and faulty cases(b).

the better disturbance attenuation and fault sensitivity performances than the results based on the diagonal Lyapunov matrices and the parameter-independent ones.

4.2. Example 2

The proposed method in this paper will be further validated by the following ship steering example which is taken from Köroğlu and Scherer (2011). Making some subtle changes and choosing the sampling period $T = 0.5$ s, we have the following discrete-time model of the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1-\omega_2 T & 0 & 0 \\ -\omega_1 k_{vr} T & 1-\omega_1 T & 0 \\ 0 & T & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_1 T \\ 0 \end{bmatrix} d(k) + \begin{bmatrix} 0.8 & 0 \\ 0.3 & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} f_1(k) \\ f_2(k) \end{bmatrix}$$
$$y(k) = x_3(k),$$

where x_1 is the sway velocity, x_3 is the yaw angle and x_2 is its rate. The dynamics depend on the velocity of the ship

$$V_s = V_0(1+\alpha), \quad \alpha \in [-\bar{\alpha}, \bar{\alpha}], \quad \bar{\alpha} = 0.25$$

through the velocity-dependent system parameters

$$k_{vr} = -0.46, \quad \omega_1 = 0.0769 V_s, \quad \omega_2 = 0.0128 V_s.$$

This system can be modelled by a two-vertex polytope with

$$\rho_1(\alpha) = \frac{\alpha + \bar{\alpha}}{2\bar{\alpha}}, \quad \rho_2(\alpha) = \frac{\bar{\alpha} - \alpha}{2\bar{\alpha}}.$$

First, solve the optimisation problem (23) with $d = 4$, $\lambda = 0.95$, $\varepsilon_1 = \varepsilon_2 = 1$. Then, assume $\alpha = 0.1$, and we can obtain

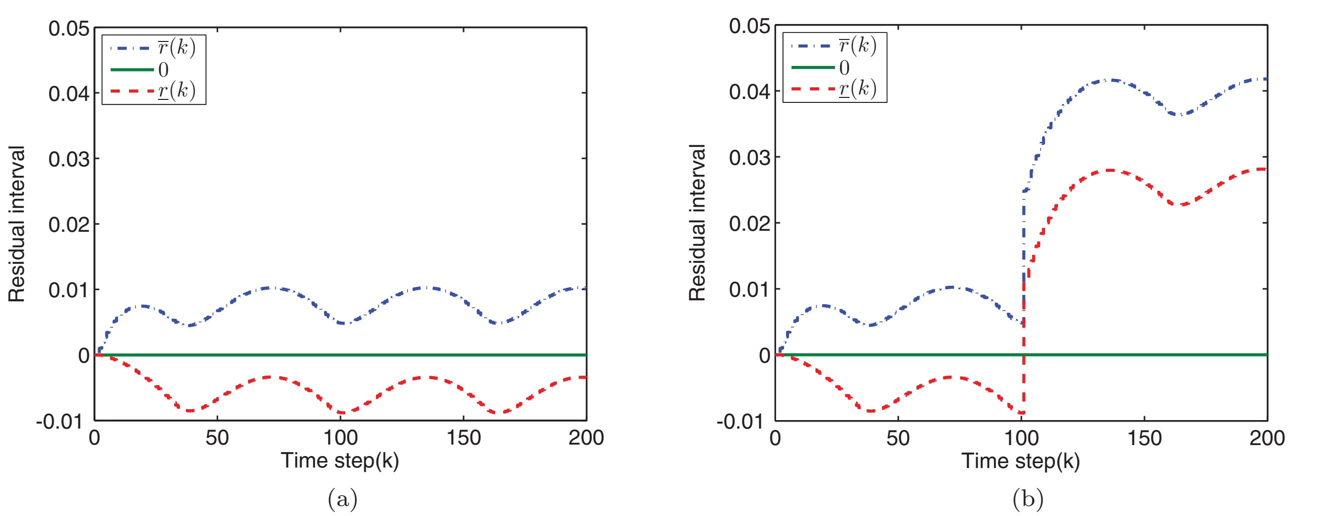

Figure 3. Residual intervals in both the fault-free (a) and faulty cases (b).

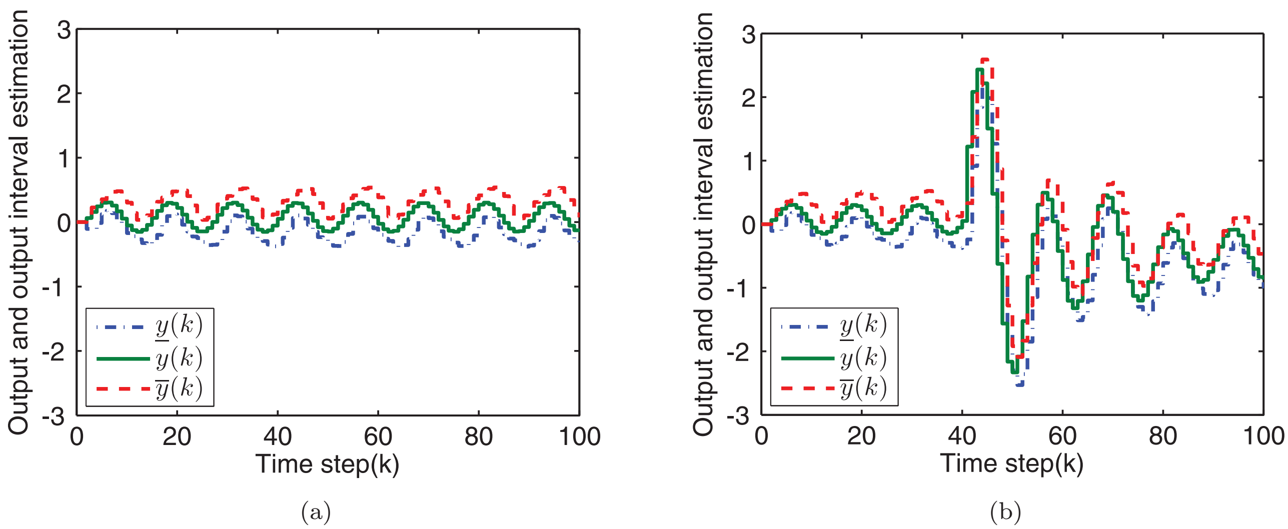

Figure 4. Outputs and output interval estimations in both the fault-free (a) and faulty cases (b).

the following interval observer gain matrices:

$$\underline{L}(\alpha) = \left(\sum_{i=1}^{2}\alpha_i W_{11i}^T\right)^{-1}\left(\sum_{i=1}^{2}\alpha_i X_i^T\right) = \left[0.0122\ -0.0140\ 0.9474\right]^T,$$

$$\overline{L}(\alpha) = \left(\sum_{i=1}^{2}\alpha_i W_{22i}^T\right)^{-1}\left(\sum_{i=1}^{2}\alpha_i Y_i^T\right) = \left[0.0098\ 0.0007\ 0.9181\right]^T,$$

$$\underline{F}(\alpha) = \left(\sum_{i=1}^{2}\alpha_i W_{11i}^T\right)^{-1}\left(\sum_{i=1}^{2}\alpha_i R_i^T\right) = \begin{bmatrix} 0.0003 & 0.0050 & 0.0237 \\ 0.0009 & 0.0017 & 0.0165 \\ 0.0119 & 0.0029 & 0.0168 \end{bmatrix},$$

$$\overline{F}(\alpha) = \left(\sum_{i=1}^{2}\alpha_i W_{22i}^T\right)^{-1}\left(\sum_{i=1}^{2}\alpha_i S_i^T\right) = \begin{bmatrix} 0.0003 & 0.0048 & 0.0204 \\ 0.0009 & 0.0016 & 0.0257 \\ 0.0108 & 0.0026 & 0.0155 \end{bmatrix},$$

What is more, we have $V_1 = \begin{bmatrix} 1.0092 \\ 1.0092 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1.0000 \\ 1.0000 \end{bmatrix}$.

Assume that the initial conditions are $\underline{x}(0) = x(0) = \overline{x}(0) = 0$, the disturbance is $d(k) = 0.3\cos(0.5k)$. The known lower and upper bounds are $\underline{d}(k) = -0.5|\cos(0.5k)|$ and $\overline{d}(k) = 0.5|\cos(0.5k)|$. The fault signal is

$$f(k) = \begin{cases} \begin{bmatrix} e^{-0.1k}(1.5\cos(0.5k) + \sin(0.5k)) \\ +2e^{-0.3k}(2\cos(2k) + 3\sin(2k)) \\ e^{-k}(1.2\cos(3k) + 2\sin(3k)) \\ +2e^{-0.1k}(2.2\cos(k) + 3.5\sin(k)) \end{bmatrix}, & k \geq 40; \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

The results are shown in Figures 4 and 5. Figure 4 shows the trajectories of system output $y(k)$, lower output estimation $\underline{y}(k)$ and upper output estimation $\overline{y}(k)$. The trajectories of lower residuals $\underline{r}_1(k)$, $\underline{r}_2(k)$ and upper residuals $\overline{r}_1(k)$, $\overline{r}_2(k)$ are shown in Figure 5. From Figure 5(a,c), it is also obvious that $0 \in [\underline{r}_1(k), \overline{r}_1(k)]$ and $0 \in [\underline{r}_2(k), \overline{r}_2(k)]$ always hold in the fault-free case. At the same time, in Figure 5(b,d), $0 \notin [\underline{r}_1(k), \overline{r}_1(k)]$ and $0 \notin [\underline{r}_2(k), \overline{r}_2(k)]$ after $k = 41$, the faults are detected almost immediately.

Besides, Table 3 illustrates not only the trade-off between the disturbance attenuation performance and the fault sensitivity performance, but also the comparison of the performances by different methods. It clearly demonstrates that the method proposed in Theorem 3.4 is less conservative.

Table 3. Comparison of the performances by different methods and cases.

	β			γ		
	Theorem 3.4	Remark 3.2	Corollary 3.1	Theorem 3.4	Remark 3.2	Corollary 3.1
$\varepsilon_1 = 10$, $\varepsilon_2 = 1$	7.5748	13.4125	7.6640	8.3832	11.4482	8.4080
$\varepsilon_1 = 1$, $\varepsilon_2 = 1$	13.0447	20.0000	13.2270	6.4297	9.3496	6.4295
$\varepsilon_1 = 1$, $\varepsilon_2 = 10$	13.6925	20.0003	13.8675	6.4037	9.3496	6.4037

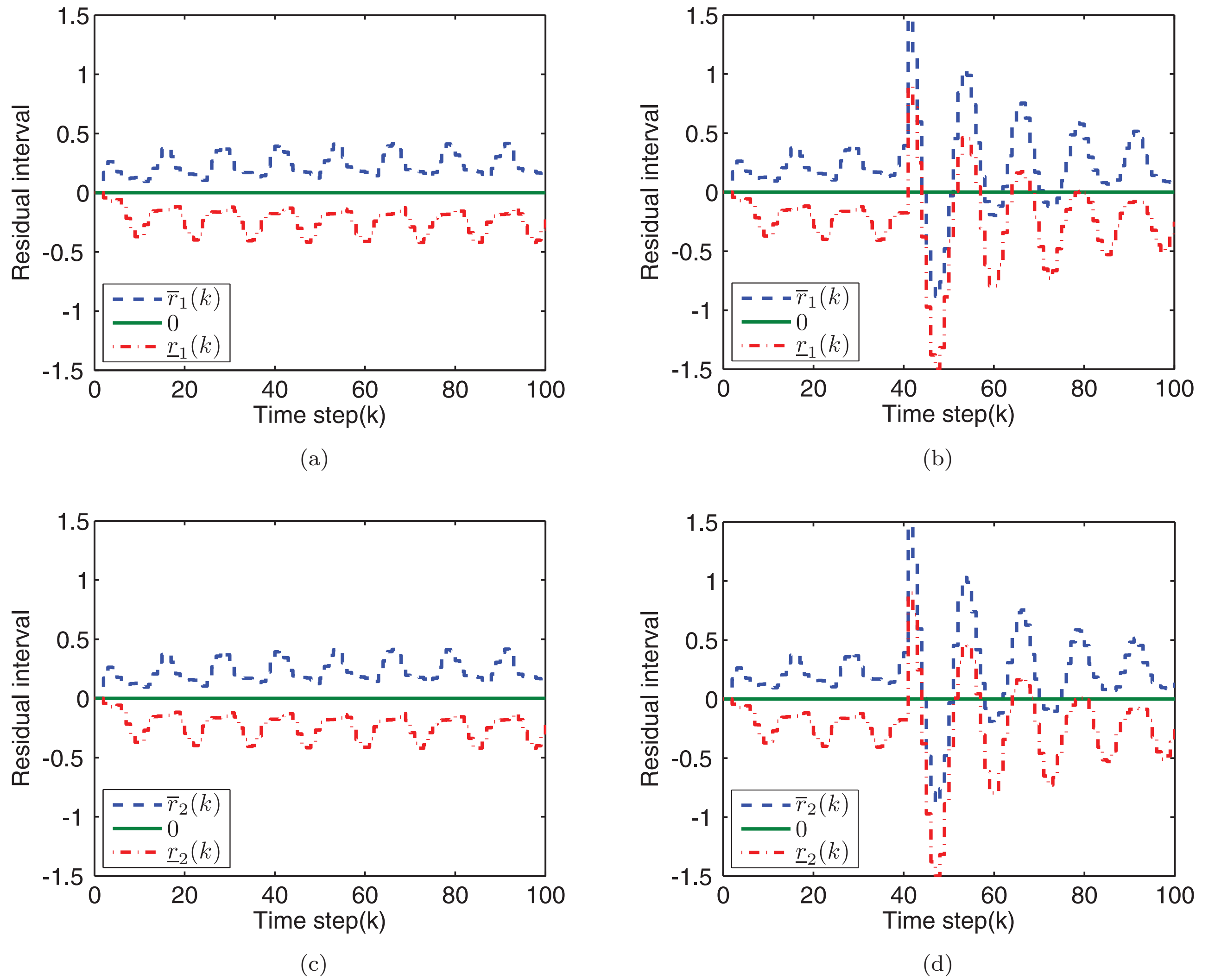

Figure 5. Residual intervals in both the fault-free (a,c) and faulty cases (b,d).

4.3. Example 3

Consider an LPV system with two parameters described by

$$\begin{aligned} x(k+1) &= A(\alpha)x(k) + B(\alpha)\omega(k) + E(\alpha)f(k) \\ y(k) &= Cx(k), \end{aligned}$$

where

$$A(\alpha) = \begin{bmatrix} -0.5 & 0 & 0 & \alpha_1 \\ 1 & 0.2 & 0 & 0.2 \\ -1 & 1 & 0.1 & -0.3 \\ 0.3 & 0 & 0.1 & -0.6+\alpha_2 \end{bmatrix},$$

$$B(\alpha) = \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \\ -0.2+\alpha_2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$E(\alpha) = \begin{bmatrix} 1 & 1 \\ \alpha_1 & 1 \\ \alpha_2 & 1 \\ 0 & 1+\alpha_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The parameters α_1 and α_2 vary according to $\alpha_1 \in [-0.5, 0.4]$ and $\alpha_2 \in [0.2, 1.2]$. This example can be modelled by a four-vertex polytope. Fixing $d = 2$, $\lambda = 0.5$, $\varepsilon_1 = \varepsilon_2 = 1$ and solving the optimisation problem (23), we can obtain the interval observer gain matrices when $\alpha_1 = 0.3$ and $\alpha_2 = 1.1$,

$$\begin{aligned} \underline{L}(\alpha) &= \left(\sum_{i=1}^{4} \alpha_i W_{11i}^T\right)^{-1} \left(\sum_{i=1}^{4} \alpha_i X_i^T\right) \\ &= \begin{bmatrix} -0.3151 & -0.0009 & 0.1842 \\ 0.8498 & 0.1991 & -0.3348 \\ -0.9855 & 0.9995 & 0.5063 \\ 0.2946 & -0.0005 & 0.1471 \end{bmatrix}, \\ \overline{L}(\alpha) &= \left(\sum_{i=1}^{4} \alpha_i W_{22i}^T\right)^{-1} \left(\sum_{i=1}^{4} \alpha_i Y_i^T\right) \\ &= \begin{bmatrix} -0.0329 & -0.0000 & -0.0201 \\ 0.9917 & 0.2010 & -0.4714 \\ -0.9586 & 1.0006 & 0.4916 \\ 0.3073 & 0.0012 & 0.1350 \end{bmatrix}, \end{aligned}$$

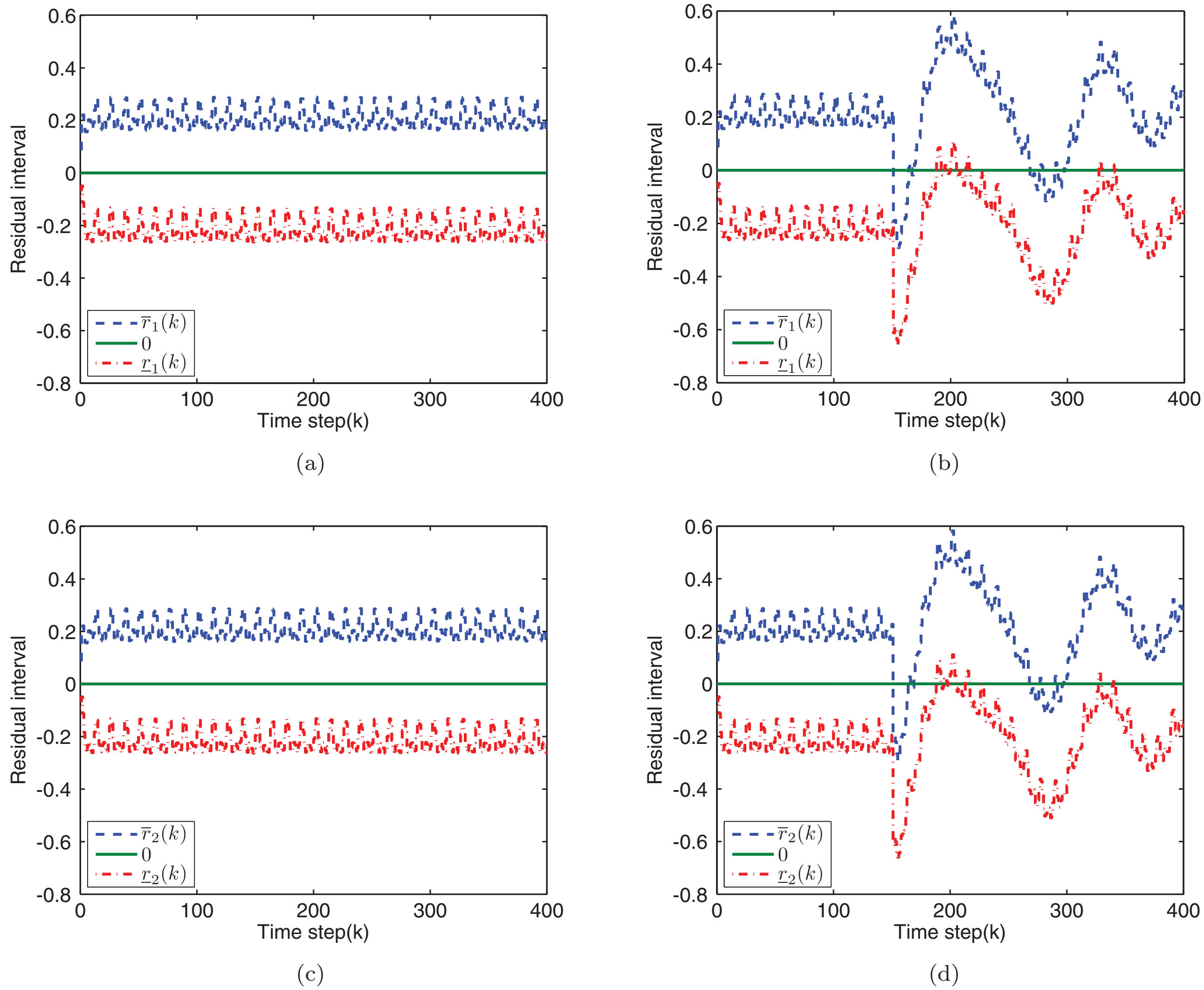

Figure 6. Residual intervals in both the fault-free (a,c) and faulty cases (b,d).

$$\underline{F}(\alpha) = \left(\sum_{i=1}^{4} \alpha_i W_{11i}^T\right)^{-1} \left(\sum_{i=1}^{4} \alpha_i R_i^T\right) = \begin{bmatrix} 0.1916 & 0.0061 & 0.1891 & 0.0060 \\ 0.0028 & 0.0060 & 0.0032 & 0.3179 \\ 0.0162 & 0.0030 & 0.4078 & 0.0016 \\ 0.0080 & 0.0037 & 0.1065 & 0.0052 \end{bmatrix},$$

$$\overline{F}(\alpha) = \left(\sum_{i=1}^{4} \alpha_i W_{22i}^T\right)^{-1} \left(\sum_{i=1}^{4} \alpha_i S_i^T\right) = \begin{bmatrix} 0.5074 & 0.0043 & 0.0021 & 0.0217 \\ 0.0020 & 0.0052 & 0.0018 & 0.3222 \\ 0.0422 & 0.0027 & 0.3926 & 0.0010 \\ 0.0091 & 0.0041 & 0.1110 & 0.0049 \end{bmatrix}.$$

Assume that the initial conditions are $\underline{x}(0) = \begin{bmatrix}-0.1 & 0.2 & -0.5 & 0.1\end{bmatrix}^T$, $x(0) = \begin{bmatrix}0.1 & 0.5 & -0.3 & 0.2\end{bmatrix}^T$ and $\overline{x}(0) = \begin{bmatrix}0.3 & 0.8 & -0.1 & 0.3\end{bmatrix}^T$, the disturbance and its bounds are $d(k) = \begin{bmatrix} 0.3\sin(k) + 0.1\cos(k) \\ 0.5\sin(0.5k) + 0.2\cos(0.5k) \end{bmatrix}$, $\underline{d}(k) = \begin{bmatrix} 0.3\sin(k) - 0.1 \\ 0.5\sin(0.5k) - 0.2 \end{bmatrix}$ and $\overline{d}(k) = \begin{bmatrix} 0.3\sin(k) + 0.1 \\ 0.5\sin(0.5k) + 0.2 \end{bmatrix}$. The fault signal is

$$f(k) = \begin{cases} \begin{bmatrix} 1.5e^{-0.03k}(0.9\cos(0.5k) \\ +\sin(0.8k)) \\ 1.4e^{-0.05k}(0.8\cos(0.6k) \\ +0.5\sin(0.4k)) \end{bmatrix}, & k \geq 150; \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

The results are shown in Figure 6. From Figure 6(b,d), $0 \notin [\underline{r}_1(k), \overline{r}_1(k)]$ and $0 \notin [\underline{r}_2(k), \overline{r}_2(k)]$ after $k = 151$, the faults can be detected successfully.

5. Conclusion

In this paper, the interval observer-based FD scheme for discrete-time LPV systems has been presented. A parameter-dependent interval observer is designed and the LMI conditions are obtained by introducing parameter-dependent Lyapunov functions and slack variables. Without the need to design extra residual evaluation functions and thresholds, the residual intervals generated by the interval observers are directly used for FD decision. The proposed method has been finally demonstrated via simulation results.

Disclosure statement

No potential conflict of interest was reported by the authors.

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